

# Two-end solutions to the Allen-Cahn equation in $\mathbb{R}^3$

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## 1 Introduction

The Allen-Cahn equation

$$-\Delta u = u - u^3, |u| < 1 \quad (1)$$

has been studied for several decades and is an important nonlinear PDE due to the fact that it lies at the interface of several different mathematical fields. The famous De Giorgi conjecture states that any entire solution to (1) in  $\mathbb{R}^n$  which is monotone in one direction should be one dimensional, at least for  $n \leq 8$ . The conjecture was proved in dimension  $n = 2$  by Ghoussoub-Gui ([13]) and dimension 3 by Ambrosio-Cabre ([1]), and in dimensions  $4 \leq n \leq 8$  by Savin ([27]), under an additional assumption. For  $n \geq 9$ , counter-examples have been constructed by del Pino-Kowalczyk-Wei ([11]). Note that monotone solutions are indeed minimizers with respect to local perturbations ([1]).

A natural extension of De Giorgi's conjecture is to classify stable or finite Morse index solutions. Regarding stable solutions, it has been shown ([1], [13]) that stable solutions in  $\mathbb{R}^2$  are necessarily one-dimensional, while Pacard-Wei ([24]) constructed stable solutions in  $\mathbb{R}^8$  which are not one-dimensional.

The study of finite Morse index solutions is much more involved. In  $\mathbb{R}^2$ , we have now a rather complete picture of Morse index one solutions. They are so-called four-end solutions, parametrized by the angle between the two lines. The cross solution, constructed by Dang-Fife-Peletier ([8]), represents a four-end solution with angle  $\frac{\pi}{4}$ , while the almost parallel line solution, constructed by del Pino-Kowalczyk-Pacard-Wei ([10]), represents a four-end solution with angle close to  $\frac{\pi}{2}$  or 0. The existence of four-end solutions with any angle between 0 and  $\frac{\pi}{2}$  was proved by two methods: the first through the moduli space theory by Kowalczyk-Liu-Pacard ([19]), and the second approach by the mountain-pass variational method by us (Gui-Liu-Wei [15]). It was also shown that the four-end solutions have Morse index one ([15]). On the other hand, monotone and symmetric properties of a general four end solution have been obtained in [14], where more general finite morse index solutions in  $\mathbb{R}^2$  have also been studied under an extra energy condition or a condition on the asymptotical structure of nodal curves.

In this paper we are interested in the structure of two-end solutions of (1) in  $\mathbb{R}^3$ . It turns out that without the monotone condition, there are actually a lot of solutions. One simple example is the so called saddle solution whose nodal sets are precisely the  $xy, yoz$  and  $xoz$  planes (Alessio-Montecchiari [3]). del Pino-Kowalczyk-Wei ([12]) proved that for each non-degenerate minimal surfaces with finite total curvature, one could find a solution to (1) whose nodal sets are close to a rescaled version of this minimal surface. In particular, there are axially solutions whose nodal sets are close to catenoids with very large waist. Axially symmetric solutions with multiple interfaces which are governed by the Jacobi-Toda system are constructed in Agudelo-del Pino-Wei [2].

In spite of all these developments, some important questions for (1) in  $\mathbb{R}^3$  remain unanswered, even for axially symmetric solutions. In this paper, we will study those axially symmetric solutions which are additionally even with respect to the  $xy$  plane. In terms of the cylindrical coordinate  $(r, z)$ , they satisfy

$$\begin{cases} u_{zz} + u_{rr} + r^{-1}u_r + u - u^3 = 0, r \in [0, +\infty), z \in \mathbb{R}, \\ u(r, z) = u(r, -z), u_r(0, z) = 0. \end{cases} \quad (2)$$

Let  $H(x) = \tanh \frac{x}{\sqrt{2}}$  be the one dimensional *heteroclinic* solution:

$$-H'' = H - H^3, H(0) = 0, H(\pm\infty) = \pm 1.$$

The solutions we are interested will have  $H$  as its asymptotic profile. We say that a solution  $u$  of (2) has growth rate  $k$  if it has the following asymptotic behavior:

$$\|u(r, \cdot) - H(\cdot - k \ln r + c)\|_{L^\infty(0, +\infty)} \rightarrow 0, \text{ as } r \rightarrow +\infty, \quad (3)$$

for certain  $c \in \mathbb{R}$ . The existence results obtained in [2] and [12] based on Lyapunov-Schmidt reduction arguments tell us that there are solutions whose growth rate is in the interval  $(\sqrt{2}, \sqrt{2} + \delta)$  and  $(\delta^{-1}, +\infty)$ , where  $\delta$  is a very small constant. A natural question is, whether or not there are solutions with

growth rate in the range  $[\sqrt{2} + \delta, \delta^{-1}]$ . In this paper we answer this question affirmatively and our main result is the following

**Theorem 1** *For each  $k \in (\sqrt{2}, +\infty)$ , there exists a solution to (2) which has growth rate  $k$ .*

As we will see later, these solutions indeed are monotone in the following sense:

$$u_z > 0 \text{ for } z > 0; u_r < 0, \text{ for } r > 0. \quad (4)$$

Outside a large ball, the nodal set of the solutions given by Theorem 1 has two components, each component is asymptotic to a catenoidal end and around each end, the solution look likes the one dimensional heteroclinic solution. Borrowing a terminology from minimal surface theory, a solution satisfying (2) and (3) will be called a two-end solution. We emphasize that here by definition the two-end solutions are axially symmetric. Comparing with the corresponding definition of two-end minimal surfaces, it seems that a more general definition of two-end solution should also involve those solutions which are not axially symmetric and only assume that their nodal set is asymptotic to two catenoidal ends. In the minimal surface theory, a classical result proved by R. Schoen ([28]) is that a minimal surface with two catenoidal ends is a catenoid. We expect that the analogous result for Allen-Cahn equation should hold. A major difficulty we encounter is to show that a solution whose nodal set is asymptotic to two catenoidal ends is axially symmetric.

Observe that for each  $k \in (0, +\infty)$ , there exists a catenoid with growth rate  $k$ . Taking into account the relation between minimal surface theory and Allen-Cahn equation, at first glance, one may think that for each  $k \in (0, +\infty)$ , there should be a two-end solution of Allen-Cahn equation. But this turns out to be false. In fact, we have

**Theorem 2** *There does not exist two-end solution with growth rate  $k \in (0, \frac{\sqrt{2}}{2}]$ .*

Indeed, one expects that for  $k \in (\frac{\sqrt{2}}{2}, \sqrt{2}]$ , two-end solution with growth rate  $k$  also should not exist, while for each  $k \in (\sqrt{2}, +\infty)$ , there should be a unique two-end solution with growth rate  $k$ . We remark that the lower bound  $\sqrt{2}$  is somehow related to the deep facts that the two ends of the solutions to the Allen-Cahn equation actually “interact” with each other and in this regime one naturally encounters the so called Toda system, as we will see later in the analysis. This constitutes a major difference with the theory of minimal surfaces. Roughly speaking, the Allen-Cahn equation interplays between the theory of minimal surfaces and the theory of Toda system, which is a classical integrable system. In the minimal surface theory, the catenoids are basic blocks for the construction of other minimal surfaces or constant mean curvature surfaces (see for example [16], [22], [29] and the references therein). It is therefore natural and interesting to ask what role two-end solutions of the Allen-Cahn equation play in constructing other solutions. We also remark that the two-end solutions given by Theorem 1 are all unstable, and the stable solution conjecture says that

the only bounded stable solution of the Allen-Cahn equation in  $\mathbb{R}^3$  should be one dimensional. It is also expected that the Morse index of two-end solutions should be equal to one, again, we don't have a proof of this statement, although it is known that the two-end solutions constructed in [2] and [12] have Morse index one.

The results in Theorem 1 could be regarded as a generalization of the corresponding results for four-end solutions in  $\mathbb{R}^2$  ([18], [19]). To explain this, let us say a few words about the multiple-end solutions in  $\mathbb{R}^2$ . By definition, a  $2k$ -end solution of (1) in  $\mathbb{R}^2$  is a solution whose nodal set outside a large ball is asymptotic to  $2k$  half straight lines at infinity. It is known (del Pino-Kowalczyk-Pacard [9]) that the set of  $2k$ -end solution in  $\mathbb{R}^2$  has a structure of real analytic variety of formal dimension  $2k$ . Some examples of solutions near the boundary of this "moduli space" have been constructed (del Pino-Kowalczyk-Pacard-Wei [10], Kowalczyk-Liu-Pacard-Wei [20]). As we mentioned above, for  $k = 2$ , it is proved in Kowalczyk-Liu-Pacard [19] that the moduli space of four-end solutions, modulo rigid motion, is diffeomorphic to the open interval  $(0, 1)$ . It is also proved there that for each  $\theta \in (0, \frac{\pi}{2})$ , there exists a four-end solution  $u_\theta$  which is even with respect to both  $x$  and  $y$  axis and the asymptotic line of the nodal set of  $u_\theta$  in the first quadrant makes angle  $\theta$  with the  $x$  axis. Now we observe that if we reflect the solution to (2) across the  $z$  axis, then we get a solution defined for all  $(r, z) \in \mathbb{R}^2$  and even in both variables. Moreover, the nodal set of this solution outside a large ball also has four components. Hence the two-end solutions in  $\mathbb{R}^3$  are in certain sense analogy of the four-end solutions in  $\mathbb{R}^2$ . The equations they satisfied are different from each other only by the term  $r^{-1}u_r$  which makes the problem inhomogeneous. In fact in  $\mathbb{R}^2$  a major fact used is that there are two linearly independent kernels  $u_x$  and  $u_y$ . The construction of four-end solution is done by first proving that all four-end solutions are *nondegenerate*. Here we face the problem of degeneracy. We overcome this difficulty by applying global bifurcation theory developed by Buffoni, Dancer and Toland for real analytical variety of formal dimension 1.

Before proceeding to proof of our main results, let us outline the main ideas of the proof, since this also gives a description of the set of solutions in Theorem 1. The main steps of the proof are similar to the corresponding analysis for four-end solution in dimension two, performed in [18] and [19]. Our basic strategy is to analyze the structure of the set  $M$  of axially symmetric two-end solutions with growth rate larger than  $\sqrt{2}$ . In section 2, we prove that the solutions in  $M$  with some additional natural constraints are compact in certain sense. In section 3, we show that  $M$  has the structure of a real analytic variety of formal dimension 1. In section 4, we analyze those solutions near the "boundary" of the moduli space  $M$  and prove that they are unique in certain sense. Combining these results, we conclude the proof by applying a structure theorem for real analytic varieties. We remark that there are two main differences between the  $\mathbb{R}^2$  and  $\mathbb{R}^3$  case. Firstly, the proof of compactness in 3D case is much more delicate than the 2D case and detailed asymptotic analysis is needed. Secondly, as we mentioned, the four-end solutions in 2D are all non-degenerate ([17], [18]), but we don't know whether it is true for two-end solutions in 3D.

## 2 Compactness of two-end solutions

The compactness of moduli spaces of minimal surface plays an important role in the minimal surface theory, for example, it is an important step towards the classification of certain type of minimal surfaces. We refer to Perez-Ros [26] and references therein for more details on this and other related subjects. In this section, we shall investigate the compactness property for two-end solutions of the Allen-Cahn equation.

Throughout the paper we denote by  $\mathbb{E}$  the set  $[0, +\infty) \times \mathbb{R}$  and by  $\mathbb{E}^+$  the set  $[0, +\infty) \times [0, +\infty)$ . Let  $\{u_n\}$  be a sequence of two-end solutions which has growth rate  $k_n > \sqrt{2}$ . (Note that for  $k \in (\frac{\sqrt{2}}{2}, \sqrt{2}]$ , we don't know whether or not there is a two-end solution with growth rate  $k$ .) Then by definition,

$$\|u_n(r, \cdot) - H(\cdot - k_n \ln r - c_n)\|_{L^\infty(0, +\infty)} \rightarrow 0, \text{ as } r \rightarrow +\infty. \quad (5)$$

We will show that if the distance of the nodal set of  $u_n$  to the origin is uniformly bounded with respect to  $n$ , then up to a subsequence,  $\{u_n\}$  converges strongly to a two-end solution  $u_\infty$ . Here converging strongly means that  $u_n \rightarrow u_\infty$  in  $C_{loc}^2(\mathbb{E})$  and there exist constants  $k_\infty, c_\infty$  such that  $k_n \rightarrow k_\infty, c_n \rightarrow c_\infty$ ,

$$\|u_\infty(r, \cdot) - H(\cdot - k_\infty \ln r - c_\infty)\|_{L^\infty(0, +\infty)} \rightarrow 0, \text{ as } r \rightarrow +\infty,$$

and

$$u_n - H(z - k_n \ln r - c_n) \rightarrow u_\infty - H(z - k_\infty \ln r - c_\infty)$$

in  $L^\infty(\mathbb{E})$ .

Under the assumption that a solution  $u$  has the asymptotic behavior (5), one could actually prove that  $u$  has certain monotonicity property.

**Lemma 3** *Suppose  $u$  is a two-end solution with growth rate  $k > \sqrt{2}$ . Then*

$$\partial_r u < 0 \text{ for } r > 0; \partial_z u > 0 \text{ for } z > 0. \quad (6)$$

The proof of this result is based on the moving plane method and its proof will be given in the appendix. By Lemma 3, the two-end solutions we are analyzing always satisfy (6). We will denote the nodal set of a solution  $u$  in the upper  $r$ - $z$  plane by

$$\mathcal{N}_u := \{p \in \mathbb{E}^+, u(p) = 0\}.$$

Due to the monotonicity property, the set  $\mathcal{N}_u \cap \partial\mathbb{E}^+$  contains a unique point, call it  $\mathcal{P}_u$ .

In the rest of the paper, we use  $C$  and  $\alpha$  to denote universal constants which may vary from step to step. The main result of this section is the following

**Proposition 4** *Let  $u_n$  be a sequence of two-end solutions with growth rate larger than  $\sqrt{2}$ . Assume*

$$|\mathcal{P}_{u_n}| \leq C.$$

*Then there exists a two-end solution  $u_\infty$  such that up to a subsequence  $\{u_n\}$  converges strongly to  $u_\infty$ .*

The rest of this section is devoted to the proof of Proposition 4.

By Lemma 3, for each solution  $u_n$ ,  $\mathcal{N}_{u_n}$  will be the graph of a function

$$z = f_n(r), r \in [\mathfrak{t}_n, +\infty).$$

Here  $\mathfrak{t}_n$  satisfies  $\mathcal{P}_{u_n} = (\mathfrak{t}_n, f_n(\mathfrak{t}_n))$ . In particular, if  $\mathcal{P}_{u_n}$  is on the  $z$  axis, then  $\mathfrak{t}_n = 0$ .

**Lemma 5** *Under the assumption of Proposition 4, we have*

$$\lim_{r \rightarrow +\infty} f_n(r) = +\infty,$$

*uniformly in  $n$ .*

**Proof.** We argue by contradiction. If this was not true, there will exist a constant  $C_0$  such that for any  $l \in \mathbb{N}$ , one could find a solution  $u_{n_l}$  satisfying

$$f_{n_l}(r) \leq C_0, \text{ for } r \in (t_{n_l}, l).$$

Since  $|u_n| < 1$ , the sequence  $\{u_{n_l}\}$  will converge in  $C_{loc}^2(\mathbb{E})$  to a nontrivial solution  $W$  whose nodal set is contained in the strip  $|z| \leq C_0$ . Moreover, by the monotonicity of  $u_n$ ,  $W_r < 0$  for  $r > 0$ . The limit  $w(z) := \lim_{r \rightarrow +\infty} W(r, z)$  then exists and is a solution of the Allen-Cahn equation in dimension 1 :

$$-w'' = w - w^3, \text{ in } \mathbb{R}. \quad (7)$$

By the symmetric property of  $W$ ,  $w(z) = w(-z)$ . We also have  $w(z) \rightarrow 1$ , as  $z \rightarrow +\infty$ . But (7) does not have a solution with nodal set containing only in a bounded set  $\{|z| \leq C_0\}$ . This is a contradiction and the proof is finished. ■

By the monotonicity of  $u_n$ ,  $\mathcal{N}_{u_n}$  is also the graph of a function over the  $z$  axis:

$$\mathcal{N}_{u_n} = \{(r, z) : r = g_n(z)\}.$$

Similar arguments with slight modification as that of (5) imply that

$$\lim_{z \rightarrow +\infty} g_n(z) \rightarrow +\infty,$$

also uniformly in  $n$ .

To obtain more information on the functions  $f_n$ , we should use the *balancing formula* ([9]). Let  $X = (0, 0, 1)$  be the constant vector field on  $\mathbb{R}^3$ . If  $u = u(x, y, z)$  is a solution to the Allen-Cahn equation, then one could check that

$$\operatorname{div} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} = 0.$$

Here  $F(u) = \frac{1}{4} (u^2 - 1)^2$ . Therefore for each regular domain  $\Omega$ , we have the following balancing formula:

$$\int_{\partial\Omega} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot v ds = 0. \quad (8)$$

Here  $v$  is the outward unit normal vector of the boundary  $\partial\Omega$ .

We would like to use (8) to control the slope of the function  $f_n$ .

**Lemma 6** For each  $\varepsilon > 0$ , there exists  $t_\varepsilon > 0$  such that for all  $n \in \mathbb{N}$ ,

$$f'_n(r_2) - f'_n(r_1) < \varepsilon \text{ for } t_\varepsilon < r_1 < r_2.$$

**Proof.** We first show that for each  $\delta > 0$  and  $\bar{r} > 0$ , there exists  $r^* > \bar{r}$ , such that

$$f'_n(r_n) < \delta, \quad n \in \mathbb{N},$$

for some  $r_n \in (\bar{r}, r^*)$ . Indeed, if this was not true, then for any  $\hat{r} > \bar{r}$ , there exists  $n$ , such that

$$f'_n(r) \geq \delta, \quad r \in (\bar{r}, \hat{r}). \quad (9)$$

Let  $\bar{r} < r_1 < r_2 < \hat{r}$ . Consider the region  $\Omega \subset \mathbb{R}^3$  given by

$$\{(x, y, z) : z > 0, L_1(r) < z < L_2(r)\},$$

where

$$\begin{aligned} L_1(r) &= f_n(r_1) - \frac{1}{f'_n(r_1)}(r - r_1), \\ L_2(r) &= f_n(r_2) - \frac{1}{f'_n(r_2)}(r - r_2). \end{aligned}$$

By the balancing formula,

$$\int_{\partial\Omega} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot v ds = 0.$$

Using (9) and the fact that  $u$  exponentially decays to 1 away from the interface, we deduce that as  $r_1 \rightarrow +\infty$ ,

$$\int_{\partial\Omega \cap \{z=0\}} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot v ds \rightarrow 0.$$

On the other hand,

$$\begin{aligned} & \int_{\partial\Omega \cap \{z=L_1(r)\}} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot v ds \\ & \sim -c_1 \frac{r_1 f'_n(r_1)}{\sqrt{1 + f'_n(r_1)^2}}, \end{aligned}$$

where  $c_1$  is a fixed constant, and

$$\begin{aligned} & \int_{\partial\Omega \cap \{z=L_2(r)\}} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot v ds \\ & \sim c_1 \frac{r_2 f'_n(r_2)}{\sqrt{1 + f'_n(r_2)^2}}. \end{aligned}$$

Combining these estimates, we get a contradiction if  $r_2$  is large enough.

Fix an  $\varepsilon > 0$ , if the conclusion of the lemma was not true, then for any  $l > 0$ , one could find  $l < r_1 < r_2$  such that

$$f'_n(r_2) - f'_n(r_1) = \varepsilon \quad (10)$$

and

$$\frac{1}{2}\varepsilon < |f'_n(r)| < \frac{3}{2}\varepsilon, r \in (r_1, r_2).$$

In this case, similarly as above, we could estimate

$$\begin{aligned} & \int_{\partial\Omega \cap \{z=0\}} \left\{ \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - (\nabla u \cdot X) \nabla u \right\} \cdot \nu ds \\ &= o(1) r_1. \end{aligned}$$

Therefore the balancing formula tells us that

$$f'_n(r_2) r_2 - f'_n(r_1) r_1 = o(1) r_1.$$

This contradicts with (10). ■

Lemma 6 tells us that the slope of the nodal line could not increase much as  $r$  increases. This in particular implies the following

**Lemma 7** *Under the assumption of Proposition 4, we have*

$$\lim_{r \rightarrow +\infty} f'_n(r) = 0,$$

*uniformly in  $n$ .*

**Proof.** We argue by contradiction and assume that there exists  $\delta > 0$  such that for each  $C_0 > 0$ , one could find  $t_0 > C_0$  and  $n$  satisfying

$$f'_n(t_0) > \delta.$$

Then one could find  $\bar{t}_1, \bar{t}_2$ , such that

$$\begin{aligned} f'_n(r) &\in \left( \frac{\delta}{2}, \delta \right), r \in (\bar{t}_1, \bar{t}_2), \\ f'_n(\bar{t}_1) &= \frac{\delta}{2}, f'_n(\bar{t}_2) = \delta. \end{aligned}$$

This contradicts with Lemma 6. ■

Since  $|u_n| < 1$ , one could assume without loss of generality that  $u_n \rightarrow u_\infty$  in  $C_{loc}^2(\mathbb{E})$  for a solution  $u_\infty$ . Note that  $u_\infty$  could not be identically 0. By the monotonicity property of  $u_n$ ,  $u_\infty$  is also monotone, that is,  $u_\infty$  satisfies (4). We need to prove  $u_n \rightarrow u_\infty$  in the strong sense, which in particular implies that  $u_\infty$  is a solution to (2) whose nodal line is asymptotic to a log curve at infinity and therefore a two-end solution.



The previous lemmas yield certain information on the nodal curve  $\mathcal{N}_{u_n}$ , but we still don't have precise description of their shape. On the other hand, we know that in the region where  $r$  is large, locally around the nodal line, the solution resembles the heteroclinic solution. Our main idea to prove the strong convergence of  $\{u_n\}$  is to get a precise decay estimate of the solution  $u_n$  to  $H$  along their nodal lines. For notational simplicity, sometimes we will drop the subscript  $n$  for the function  $u_n, f_n$ .

To proceed, let us introduce the *Fermi coordinate* around the smooth curve  $z = f(\cdot)$ . This coordinate, denoted by  $(r_1, z_1)$ , is defined through the relation

$$(r, z) = (r_1, f(r_1)) + z_1 \mathbf{n},$$

where  $\mathbf{n}$  is the unit normal vector at the point  $(r_1, f(r_1))$  of the curve  $z = f(\cdot)$ , upward pointed. Explicitly,

$$\begin{cases} r = r_1 - z_1 \frac{f'}{\sqrt{1+f'^2}}, \\ z = f + z_1 \frac{1}{\sqrt{1+f'^2}}. \end{cases} \quad (11)$$

Here the function  $f$  is evaluated at the point  $r_1$ . The map  $X : (r_1, z_1) \rightarrow (r, z)$  from the Fermi coordinate to the original coordinate system will be a diffeomorphism in certain tubular neighborhood of the graph of  $f$ .

From (11), we get

$$\begin{aligned} \begin{pmatrix} \partial_r r_1 & \partial_z r_1 \\ \partial_r z_1 & \partial_z z_1 \end{pmatrix} &= \begin{pmatrix} \partial_{r_1} r & \partial_{z_1} r \\ \partial_{r_1} z & \partial_{z_1} z \end{pmatrix}^{-1} \\ &= \begin{pmatrix} B & -\frac{f'}{\sqrt{1+f'^2}} \\ f'B & \frac{1}{\sqrt{1+f'^2}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{(1+f'^2)B} & \frac{f'}{(1+f'^2)B} \\ -\frac{f'}{\sqrt{1+f'^2}} & \frac{1}{\sqrt{1+f'^2}} \end{pmatrix}. \end{aligned} \quad (12)$$

Here

$$B = 1 - \frac{z_1 f''}{(1+f'^2)^{\frac{3}{2}}}.$$

Let us denote by  $\Delta_{(r,z)} = \partial_r^2 + \partial_z^2$  and  $\Delta_{(r_1,z_1)} = \partial_{r_1}^2 + \partial_{z_1}^2$ . These two operators are related by

$$\begin{aligned} \Delta_{(r,z)} &= \Delta_{(r_1,z_1)} + \left( \frac{1}{A} - 1 \right) \partial_{r_1}^2 \\ &\quad + \frac{1}{2} \frac{\partial_{z_1} A}{A} \partial_{z_1} - \frac{1}{2} \frac{\partial_{r_1} A}{A^2} \partial_{r_1}, \end{aligned} \quad (13)$$

where

$$A = 1 + f'^2 - 2z_1 \frac{f''}{\sqrt{1+f'^2}} + z_1^2 \frac{f''^2}{(1+f'^2)^2}. \quad (14)$$

This follows from direct computation, see [10] for more details. From formula (13), one sees that if  $f', f'', f^{(3)}$  is small, then  $\Delta_{(r,z)}$  is a small perturbation of  $\Delta_{(r_1, z_1)}$ . One should keep in mind that in (13), there are terms (in  $\partial_{r_1} A$ ) involving the third derivatives of  $f$  and these terms should be dealt with very carefully.

For any function  $\Theta(r, z)$ , the pullback of  $\Theta$  by the diffeomorphism  $X$  will be defined as

$$X^* \Theta(r_1, z_1) := \Theta \circ X(r_1, z_1).$$

Occasionally,  $X^* \Theta$  will also be denoted by  $\Theta^*$ . Keep in mind that  $X^* \Theta$  is only defined in the region where the Fermi coordinate is well defined. Conversely, for any function  $\Theta(r_1, z_1)$ , we set  $(X^{-1})^* \Theta(r, z) = \Theta \circ X^{-1}(r, z)$ .

To prove the compactness, we need to know the precise asymptotic behavior of  $u_n$ . We shall define suitable approximate solutions and analyze the error between the approximate and true solutions. It turns out that to get a good approximate solution, it will be convenient to use the Fermi coordinate. Clearly there is a technical issue concerning the Fermi coordinate. Namely, the Fermi coordinate is in general not well defined in the whole  $r$ - $z$  plane. Note that for  $r_1$  large, by Lemma 7,  $f'$  is small. We also know that  $u$  is close to the one dimensional heteroclinic solution locally around the nodal line at far away. Hence intuitively, for  $r$  large,  $f'', f^{(3)}, \dots$ , should also be small. Indeed, we have

**Lemma 8** *As  $r$  tends to infinity,  $|f''(r)|, |f^{(3)}(r)|, |f^{(4)}(r)|$  tend to zero.*

**Proof.** Let  $r_0$  be a fixed large constant. Around the point  $(r_0, f(r_0))$ , let us consider the line  $l_1$

$$z - f(r_0) = f'(r_0)(r - r_0).$$

Let  $l_2$  be the line orthogonal to it and passing through  $(r_0, f(r_0))$ . Use these two lines to build an orthogonal coordinate system, denoted by  $(s, t)$ , where the  $s$  coordinate corresponding to line  $l_1$ . We know that  $u$  is close to  $H(t)$ . By analyzing the equation satisfied by the error  $\phi(r, z) := u(r, z) - H(t)$ , we find that for  $r_0$  large, around  $(r_0, f(r_0))$ , the  $C^2$  norm of  $\phi$  is small.

Observe that  $u(r, f(r)) = 0$ . Hence using the definition of  $\phi$ ,

$$H\left(\frac{f(r) - [f(r_0) + f'(r_0)(r - r_0)]}{\sqrt{1 + (f'(r_0))^2}}\right) + \phi(r, f(r)) = 0. \quad (15)$$

Differentiate this equation with respect to  $r$ , using the fact that  $\partial_z \phi$  is small, we get  $|f'(r) - f'(r_0)|$  is small, which we already know. Differentiate (15) twice, we obtain

$$H'(\cdot) f''(r) + H''(\cdot) (f'(r) - f'(r_0))^2 + \partial_r^2 \phi + 2\partial_r \partial_z \phi f' + \partial_z \phi f'' = 0.$$

Using the fact that  $\partial_z \phi$  is small, we deduce  $f''(r_0)$  is small.

Indeed, one could continue this process and show that the higher order derivatives  $f^{(3)}, f^{(4)}$  are also small. ■

Since  $|f''|$  is small, the Fermi coordinate actually will be well defined in a very large tubular neighborhood of  $\mathcal{N}_u$ . For later purposes, we need to be more precise in describing the size of the region where Fermi coordinate is well defined.

For each point  $Z = (r, z) \in \mathbb{E}$ , we use  $\text{dist}(Z, \mathcal{N}_u)$  to denote the distance between  $Z$  and the curve  $\mathcal{N}_u$ . That is,

$$\text{dist}(Z, \mathcal{N}_u) = \min \{|Z - p|, p \in \mathcal{N}_u\}.$$

Let  $\pi(Z)$  be the set of points which realize this distance. Hence

$$\pi(Z) = \{p \in \mathcal{N}_u : |Z - p| = \text{dist}(Z, \mathcal{N}_u)\}.$$

For each  $r_1$ , let us consider the set

$$D_{r_1} := \{Z : \pi(Z) = \{(r_1, f(r_1))\}\}.$$

Note that by the triangle inequality,  $D_{r_1}$  is connected. Then

$$D_{r_1} = \{X(r_1, t) : t \in (-d_1(r_1), d_2(r_1))\},$$

for some functions  $d_1$  and  $d_2$  ( $d_i$  may not be differentiable).

Define

$$\bar{d}(r_1) = \min \{d_1(r_1), d_2(r_1), 3f(r_1)\}$$

and

$$\bar{D}_{r_1} = \{X(r_1, t) : t \in (-\bar{d}(r_1), \bar{d}(r_1))\}.$$

Here the constant 3 has not particular importance. Fix a large constant  $r_0$ . Let  $\mathcal{B}_u := \cup_{r_1 > r_0} \bar{D}_{r_1}$ . It is worth to be pointed out that in principle the boundary of  $\mathcal{B}_u$  could be quite complicated. Let  $\eta$ , whose existence is related to assumption (A) below, be a smooth cutoff function equals 0 outside  $\cup_{r_1 > r_0-1} \bar{D}_{r_1}$  and

$$X^* \eta(r_1, z_1) = 1, \text{ for } z_1 \in (-\bar{d}(r_1) + 1, \bar{d}(r_1) - 1), r_1 > r_0.$$

Similarly, let  $\mathcal{B}_u^+ = \mathcal{B}_u \cap \mathbb{E}^+$  and we define a similar cutoff function  $\eta^+$  supported in  $\mathcal{B}_u^+$  (note that the notation  $\eta^+$  does not mean the positive part of  $\eta$ ).

Since  $\mathcal{B}_u$  is not the whole  $r$ - $z$  plane, we shall use the cutoff function  $\eta$  to define a smooth function  $\mathcal{H}_1$  by the formula

$$\mathcal{H}_1 = \eta H_1 + (1 - \eta) \frac{H_1}{|H_1|}.$$

Here the function  $H_1$  is defined through

$$X^* H_1(r_1, z_1) = H(r_1 - h(z_1)),$$

where  $h$  is a small function to be determined later. We emphasize that since we are only interested in the behavior of  $u$  outside of a bounded set,  $\mathcal{H}_1$  should be regarded as a function only defined on the set  $\hat{\mathbb{E}}$ , where

$$\hat{\mathbb{E}} = \mathbb{E} \setminus \left\{ (r, z) : -f(r_0) + \frac{1}{f'(r_0)}(r - r_0) < z < f(r_0) - \frac{1}{f'(r_0)}(r - r_0) \right\}.$$

For any function  $\Theta$ , let  $\Theta^s(r, z) = \Theta(r, -z)$ .

With all the previous notations introduced, we now define an approximate solution  $\bar{u}$  as

$$\bar{u} = \mathcal{H}_1 + \mathcal{H}_1^s + 1.$$

The function  $\bar{u}$  is defined on the set  $\hat{\mathbb{E}}$ , rather than the whole plane  $\mathbb{E}$ . Let  $\phi = u - \bar{u}$  be the difference between the true solution  $u$  and the approximate solution  $\bar{u}$ . Note that since  $\bar{u}$  is even with respect to the  $z$  variable,  $\phi$  is also even. Introduce the function  $H'_1$  by

$$X^* H'_1(r_1, z_1) = H'(z_1 - h(r_1)).$$

Then the small function  $h$  appeared in the definition of the function  $H_1$  is required to satisfy the following orthogonality condition:

$$\int_{\mathbb{R}} X^* (\phi \eta^+ H'_1) dz_1 = 0, \quad r_1 > r_0. \quad (16)$$

The existence of  $h$  is guaranteed by the following:

**Lemma 9** *There exists a small function  $h$  (in  $C^2$  sense) satisfying (16).*

**Proof.** This follows from similar arguments as that of [19]. The basic idea is to use the fact that  $\phi$  is small and apply the implicit function theorem. We omit the details. ■

One advantage of using the Fermi coordinate with respect to the nodal curve is that  $h$  could be estimate in terms of  $\phi$  and  $f$ . To see this, let  $\mathcal{N}^s(u)$  be the graph of the function  $z = -f(r)$ . For  $Z = (r, z) \in \mathbb{E}$ , let

$$\mathcal{D}(r, z) = \text{dist}(Z, \mathcal{N}(u)) + \text{dist}(Z, \mathcal{N}^s(u)),$$

that is, the sum of the distance of  $Z$  to the nodal line in the upper plane and lower plane. Set  $D(r_1) = \mathcal{D}(r_1, f(r_1))$ .

**Lemma 10** *For  $r_1 > r_0$ ,*

$$|h(r_1)| \leq C |X^* \phi(r_1, 0)| + C e^{-\sqrt{2}D(r_1)}.$$

**Proof.** Since  $\phi = u - \bar{u}$ , around  $\mathcal{N}_u$  we have

$$X^* \phi(r_1, z_1) = X^* u(r_1, z_1) - [H(z_1 - h(r_1)) + X^* \mathcal{H}_1^s + 1]. \quad (17)$$

Setting  $z_1 = 0$  in the above equation, using the fact that  $X^* u(r_1, 0) = 0$ , we find

$$X^* \phi(r_1, 0) + H(-h(r_1)) + 1 + X^* \mathcal{H}_1^s(r_1, 0) = 0.$$

On the other hand, by the definition of  $\mathcal{H}_1^s$  and the asymptotic behavior of  $H$ ,

$$|1 + X^* \mathcal{H}_1^s(r_1, 0)| \leq C e^{-\sqrt{2}D(r_1)}.$$

Therefore,

$$|h(r_1)| \leq C |X^* \phi(r_1, 0)| + C e^{-\sqrt{2}D(r_1)}.$$

This completes the proof. ■

Following the above arguments and differentiate equation (17) with respect to  $r_1$  and setting  $z_1 = 0$  in the obtained equation, we get

$$|h'(r_1)| \leq C |\partial_{r_1} X^* \phi(r_1, 0)| + C e^{-\sqrt{2}D(r_1)},$$

Similar arguments yield estimates for the higher order derivatives and Holder norms.

One of the main ingredients in the proof of Proposition 4 is to get suitable decay estimate for the function  $\phi$ . In this respect, we shall first of all prove the following estimate.

**Proposition 11** *For  $r_1 > r_0$ , the function  $\phi$  satisfies*

$$\|X^* \phi(r_1, \cdot)\|_{L^\infty} \leq C e^{-r_1} + C e^{-D(r_1)} + \frac{C}{r_1^2}.$$

We shall first of all prove this proposition under an additional assumption on the size of the Fermi coordinate. More precisely, at this stage, we assume that

$$|\bar{d}'| \leq \frac{1}{2}, |\bar{d}''| \leq C \text{ and } \bar{d}(r) \geq \frac{2D(r)}{3}. \quad (\text{A})$$

Later we will indicate the necessary modification of the proof if this assumption is not a priori satisfied.

Clearly  $\phi$  satisfies

$$L\phi := -\Delta\phi + (3\bar{u}^2 - 1)\phi = E(\bar{u}) + P(\phi). \quad (18)$$

Here the notation  $E(\bar{u})$  stands for

$$\bar{u}_{rr} + r^{-1}\bar{u}_r + \bar{u}_{zz} + \bar{u} - \bar{u}^3,$$

which is the error of the approximate solution  $\bar{u}$ , and

$$P(\phi) = -3\bar{u}\phi^2 - \phi^3$$

is a higher order perturbation term.

For any function  $\Theta$ , we define the projection of  $\Theta$  onto  $\eta^+ H'_1$  as

$$\Theta_1^\parallel = \frac{\int_{\mathbb{R}} X^* (\Theta \eta^+ H'_1) dz_1}{\int_{\mathbb{R}} X^* ((\eta^+)^2 H_1'^2) dz_1} \eta^+ H'_1.$$

Since the solutions are even in the  $z$  variable, we also define

$$\Theta^\parallel = \Theta_1^\parallel + \left[ \Theta_1^\parallel \right]^s.$$

Finally, set

$$\Theta^\perp := \Theta - \Theta^\parallel.$$

Equation (18) could be written in the form

$$L\phi = [E(\bar{u})]^\perp + [E(\bar{u})]^\parallel + P(\phi).$$

As we will see later,  $[E(\bar{u})]^\parallel$  is small (in suitable norm) compared to  $\phi$ , and  $\phi$  is then essentially controlled by  $[E(\bar{u})]^\perp$ . As a crucial step towards the proof of Proposition 11, we shall take the task of analyzing  $E(\bar{u})$ . By definition

$$\bar{u} = \mathcal{H}_1 + \mathcal{H}_1^s + 1.$$

A simple manipulation leads to the following expansion

$$E(\bar{u}) = E(\mathcal{H}_1) + E(\mathcal{H}_1^s) + 3(\mathcal{H}_1^s + 1)(\mathcal{H}_1^2 - 1) + 3(\mathcal{H}_1 + 1)(\mathcal{H}_1^s + 1)^2. \quad (19)$$

Certainly the function  $E(\bar{u})$  is also even in the  $z$  variable. One expects that in the upper plane  $\mathbb{E}^+$ , the main order of  $E(\bar{u})$  should be  $E(\mathcal{H}_1)$  and one of the interaction term  $3(\mathcal{H}_1^s + 1)(\mathcal{H}_1^2 - 1)$ .

We first analyze the projection of  $E(\mathcal{H}_1)$  onto  $\eta^+ H_1'$ . For technical reasons, we introduce a small perturbation of  $f$  due to the presence of the modulation function  $h$ . Let  $p = f + \hat{h}$ , where  $\hat{h} := \sqrt{1 + f'^2}h$ . Throughout the paper, we set  $\mathbf{c}_0 = \int_{\mathbb{R}} H'^2(t) dt$ . The notation  $O(g)$  represents a function in  $r_1$  such that  $|O(g)| \leq C|g(r_1)|$ .

**Lemma 12** For  $r_1 > r_0$ ,

$$\begin{aligned} \int_{\mathbb{R}} X^* (\eta^+ H_1' E(\mathcal{H}_1)) dz_1 &= (1 + o(1)) \frac{\mathbf{c}_0}{r_1} \left( \frac{r_1 p'}{\sqrt{1 + p'^2}} \right)' + O(h'' f'') + O(h'^2) \\ &+ O(h' f'') + O(h' r^{-1}) + O(e^{-2\sqrt{2}\bar{d}(r_1)}) + O(h' h'') + O(h f'') \\ &+ O(h f''') + O(f'^3) + O\left(\frac{f'^3}{r_1^3}\right). \end{aligned}$$

**Proof.** In the region where the cutoff function  $\eta^+ \neq 0$ ,  $E(\mathcal{H}_1)$  could be expressed in terms of the Fermi coordinate  $(r_1, z_1)$ :

$$\begin{aligned} E(\mathcal{H}_1) &= \Delta_{(r,z)} H_1 + r^{-1} \partial_r H_1 + H_1 - H_1^3 \\ &= A^{-1} \partial_{r_1}^2 H_1 + \partial_{z_1}^2 H_1 \\ &+ \frac{\partial_{z_1} A}{2A} \partial_{z_1} H_1 - \frac{\partial_{r_1} A}{2A^2} \partial_{r_1} H_1 \\ &+ r^{-1} (\partial_{r_1} H_1 \partial_r r_1 + \partial_{z_1} H_1 \partial_r z_1) + H_1 - H_1^3. \end{aligned}$$

Obviously  $\partial_{z_1} H_1 = H'$ ,  $\partial_{z_1}^2 H_1 = H''$ , and

$$\partial_{r_1} H_1 = -h' H', \partial_{r_1}^2 H_1 = -h'' H' + h'^2 H'',$$

where  $H'$  and  $H''$  are evaluated at  $z_1 - h(r_1)$ . Since  $H'' + H - H^3 = 0$ , we obtain

$$\begin{aligned} E(\mathcal{H}_1) &= A^{-1}(-h''H' + h'^2H'') + \left(\frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1\right)H' \\ &\quad + \left(-\frac{\partial_{r_1}A}{2A^2} + r^{-1}\partial_r r_1\right)(-h'H'). \end{aligned} \quad (20)$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}} X^*(\eta^+ H'_1 E(\mathcal{H}_1)) dz_1 \\ &= \int_{\mathbb{R}} \eta^{+*} A^{-1}(-h''H' + h'^2H'') H' + \int_{\mathbb{R}} \eta^{+*} \left(\frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1\right) H'^2 \\ &\quad + \int_{\mathbb{R}} \eta^{+*} \left(\frac{\partial_{r_1}A}{2A^2} - r^{-1}\partial_r r_1\right) h'H'^2 + O(e^{-2\sqrt{2}d}) \\ &= -h'' \int_{\mathbb{R}} \eta^{+*} A^{-1} H'^2 + \int_{\mathbb{R}} \eta^{+*} \left(\frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1\right) H'^2 \\ &\quad + O(h'^2) + O(h'f'') + O(h'f''') + O(h'r^{-1}) + O(e^{-2\sqrt{2}d}). \end{aligned}$$

The appearing of the term  $O(e^{-2\sqrt{2}d})$  is due to the fact that in the upper boundary of  $\mathcal{B}_u$ ,  $E(\mathcal{H}_1) = O(e^{-\sqrt{2}d})$ .

To proceed, we shall calculate the second term in the above expression, which roughly speaking should be the main order term of the projection. We have

$$\begin{aligned} &\frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1 \\ &= \frac{-\frac{f''}{\sqrt{1+f'^2}} + z_1 \frac{f''^2}{(1+f'^2)^2}}{A} - \frac{f'}{\left(r_1 - z_1 \frac{f'}{\sqrt{1+f'^2}}\right) \sqrt{1+f'^2}} \\ &= \left(-\frac{f''}{(1+f'^2)^{\frac{3}{2}}} + z_1 \frac{f''^2}{(1+f'^2)^3}\right) \left(1 + 2z_1 \frac{f''}{(1+f'^2)^{\frac{3}{2}}} + O(f''^2)\right) \\ &\quad - \frac{f'}{r_1 \sqrt{1+f'^2}} \left(1 + z_1 \frac{f'}{r_1 \sqrt{1+f'^2}} + O\left(\frac{f'^2}{r_1^2}\right)\right) \\ &= -\frac{f''}{(1+f'^2)^{\frac{3}{2}}} - \frac{z_1 f''^2}{(1+f'^2)^3} + O(f''^3) \\ &\quad - \frac{f'}{r_1 \sqrt{1+f'^2}} - \frac{z_1 f'^2}{r_1^2 (1+f'^2)} + O\left(\frac{f'^3}{r_1^3}\right). \end{aligned}$$

Using this estimate, we get

$$\begin{aligned}
& \int_{\mathbb{R}} X^* (\eta^+ H'_1 E(\mathcal{H}_1)) \\
&= - (1 + o(1)) \left[ \frac{\mathbf{c}_0 h''}{1 + f'^2} + \frac{\mathbf{c}_0 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 f'}{r_1 \sqrt{1 + f'^2}} \right] \\
&+ O(h'^2) + O(h' f'') + O(h' f''') + O(h' r^{-1}) \\
&+ O(h'' f'') + O\left(e^{-2\sqrt{2}\bar{d}(r_1)}\right).
\end{aligned} \tag{21}$$

In this expression, we are mainly interested in the first term. We calculate

$$\begin{aligned}
& \frac{r_1 h''}{1 + f'^2} + \frac{r_1 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{f'}{\sqrt{1 + f'^2}} \\
&= \frac{r_1}{1 + f'^2} \left[ \frac{\hat{h}''}{\sqrt{1 + f'^2}} + 2\hat{h}' \left( \frac{1}{\sqrt{1 + f'^2}} \right)' + \hat{h} \left( \frac{1}{\sqrt{1 + f'^2}} \right)'' \right] \\
&+ \frac{r_1 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{f'}{\sqrt{1 + f'^2}} \\
&= \left( \frac{r_1 p'}{\sqrt{1 + p'^2}} \right)' + O(h' r_1 h'') + O(h' r_1 f'') \\
&+ O(h r_1 f'') + O(h r_1 f''') + O(h').
\end{aligned}$$

The conclusion of the lemma then follows from this estimate. ■

With the projection being understood, we proceed to analyze the orthogonal part.

**Lemma 13** *For  $r_1 > r_0$ , the following estimate is true:*

$$\begin{aligned}
E(\mathcal{H}_1) - [E(\mathcal{H}_1)]_1^\parallel &= -(z_1 - h) \left( \frac{f''^2}{(1 + f'^2)^3} + \frac{f'^2}{r_1^2 (1 + f'^2)} \right) H' \\
&+ O(h'' f'') + O\left(e^{-\sqrt{2}D}\right) + O(h' r^{-1}) + O(h''^2) \\
&+ O(f''^3) + O\left(\frac{f'^3}{r_1^3}\right) + O(h'^2) + O(h' f'') + O(h' f''').
\end{aligned} \tag{22}$$



**Proof.** Let us consider typical terms appeared in (20).

$$\begin{aligned}
& A^{-1}h''H' - [A^{-1}h''H']_1^\parallel \\
&= A^{-1}h''H' - \frac{\int_{\mathbb{R}} A^{-1}h''\eta^{+*}H'^2}{\int_{\mathbb{R}} (\eta^{+*}H')^2} \eta^{+*}H' \\
&= \frac{h''H'}{1+f'^2} \left[ 1 - \frac{\eta^{+*}}{\int_{\mathbb{R}} (\eta^{+*}H')^2} \int_{\mathbb{R}} \eta^{+*}H'^2 \right] + O(f''h'') \\
&= O(h'') \left( \int_{\mathbb{R}} (\eta^{+*}H')^2 - \eta^{+*} \int_{\mathbb{R}} \eta^{+*}H'^2 \right) H' + O(f''h'').
\end{aligned}$$

Note that

$$\left( \int_{\mathbb{R}} (\eta^{+*}H')^2 - \eta^{+*} \int_{\mathbb{R}} \eta^{+*}H'^2 \right) H' = O(e^{-\sqrt{2}f}).$$

Hence

$$A^{-1}h''H' - [A^{-1}h''H']_1^\parallel = O(h'^2) + O(e^{-\sqrt{2}D}) + O(f''h'').$$

Next, we have calculated that

$$\begin{aligned}
\frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1 &= -\frac{f''}{(1+f'^2)^{\frac{3}{2}}} - \frac{z_1 f'^2}{(1+f'^2)^3} \\
&\quad - \frac{f'}{r_1 \sqrt{1+f'^2}} - \frac{z_1 f'^2}{r_1^2 (1+f'^2)} \\
&\quad + O(f'^3) + O\left(\frac{f'^3}{r_1^3}\right).
\end{aligned}$$

Using this expansion, we find

$$\begin{aligned}
& \left( \frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1 \right) H' - \left[ \left( \frac{\partial_{z_1}A}{2A} + r^{-1}\partial_r z_1 \right) H' \right]_1^\parallel \\
&= -(z_1 - h) \left( \frac{f'^2}{(1+f'^2)^3} + \frac{f'^2}{r_1^2 (1+f'^2)} \right) H' \\
&\quad + O(f'^3) + O\left(\frac{f'^3}{r_1^3}\right) + O(e^{-\sqrt{2}D}).
\end{aligned}$$

The rest of the terms always contains small order terms of  $h'$  or  $h$  and the conclusion of the lemma readily follows. ■

The next result gives us an estimate for the main interaction term between  $\mathcal{H}_1$  and  $\mathcal{H}_1^s$  appeared in  $E(\bar{u})$ . Let

$$\mathbf{c}_1 = 3\sqrt{2} \int_{\mathbb{R}} H'^2(s) e^{\sqrt{2}s} ds.$$

**Lemma 14** For  $r_1 > r_0$ ,

$$\int_{\mathbb{R}} X^* [3\eta^+ H_1' (\mathcal{H}_1^s + 1) (\mathcal{H}_1^2 - 1)] dz_1 = -\mathbf{c}_1 (1 + o(1)) e^{-\sqrt{2}D(r_1)}.$$

**Proof.** Since  $H'' = H^3 - H$ ,  $H_1^2 - 1 = -\sqrt{2}H_1'$ , it follows that

$$\begin{aligned} & \int_{\mathbb{R}} X^* [3\eta^+ H_1' (\mathcal{H}_1^s + 1) (\mathcal{H}_1^2 - 1)] dz_1 \\ &= -3\sqrt{2} \int_{\mathbb{R}} X^* [\eta^+ H_1'^2 (\mathcal{H}_1^s + 1)] dz_1 + o\left(e^{-\sqrt{2}D}\right) \\ &= -3\sqrt{2} \int_{\mathbb{R}} X^* (\eta^+ H_1'^2) e^{-\sqrt{2}|z_2|} dz_1 \\ &+ O\left(\int_{\mathbb{R}} X^* (\eta^+ H_1'^2) e^{-2\sqrt{2}|z_2|} dz_1\right). \end{aligned}$$

Since by definition  $|z_2| = \mathcal{D}(r, z) - |z_1|$ , we find

$$\begin{aligned} & \int_{\mathbb{R}} X^* [3\eta^+ H_1' (\mathcal{H}_1^s + 1) (\mathcal{H}_1^2 - 1)] dz_1 \\ &= -3\sqrt{2} \int_{\mathbb{R}} X^* (\eta^+ H_1'^2) e^{-\sqrt{2}(X^*\mathcal{D}-|z_1|)} dz_1 \\ &+ O\left(\int_{\mathbb{R}} X^* (\eta^+ H_1'^2) e^{-2\sqrt{2}(X^*\mathcal{D}-|z_1|)} dz_1\right) \\ &= -3\sqrt{2} \int_{\mathbb{R}} X^* (\eta^+ H_1'^2) e^{-\sqrt{2}(X^*\mathcal{D}-|z_1|)} dz_1 \\ &+ o\left(e^{-\sqrt{2}D(r_1)}\right) \\ &= -3\sqrt{2}e^{-\sqrt{2}D(r_1)} \int_{\mathbb{R}} X^* (H_1'^2) e^{\sqrt{2}|z_1|} dz_1 \\ &+ o\left(e^{-\sqrt{2}D(r_1)}\right). \end{aligned}$$

In the last equality, we have used the fact that in  $\mathbb{E}^+$ ,

$$|X^*\mathcal{D}(r_1, z_1) - D(r_1)| = o(z_1).$$

■

**Remark 15** Clearly the function  $D(r_1)$  is strictly less than  $2f(r_1)$ . However, due to the fact that  $f'$  is small,

$$2f(r_1) - D(r_1) = o(f(r_1)).$$

It is exactly this fact that makes it possible for us to analyze the equation satisfied by  $f$ .

The previous result analyze directly the error  $E(\bar{u})$  from the definition of  $\bar{u}$ . In the next result, we will express the projection of  $E(\bar{u})$  onto  $\eta^+ H'_1$  in terms of the function  $\phi$ . The main equation we will use is equation (18) satisfied by  $\phi$ .

**Lemma 16** *For each  $r_1 > r_0$ ,*

$$\begin{aligned} & \left| \int_{\mathbb{R}} X^* (\eta^+ H'_1 E(\bar{u})) dz_1 \right| \\ &= O \left( \|\phi^* (r_1, \cdot)\|_{\infty}^2 \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)} \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right) \\ &+ O \left( \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)} \right) + O \left( \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty}^2 \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} \|\partial_{r_1}^2 \phi^* (r_1, \cdot)\|_{\infty} \right) \\ &+ O \left( f'' \|\partial_{z_1} \phi^* (r_1, \cdot)\|_{\infty} \right) + O \left( f'' \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right) + O \left( f''' \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right) \\ &+ O \left( r^{-1} \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right) + O \left( r^{-1} f' \|\partial_{z_1} \phi^* (r_1, \cdot)\|_{\infty} \right). \end{aligned}$$

**Proof.** Multiplying both sides of (18) with  $\eta^+ H'_1$  and integrating in  $\mathbb{R}$ , we get

$$\int_{\mathbb{R}} X^* (\eta^+ H'_1 E(\bar{u})) dz_1 = \int_{\mathbb{R}} X^* (\eta^+ H'_1 L\phi) dz_1 - \int_{\mathbb{R}} X^* (\eta^+ H'_1 P(\phi)) dz_1.$$

Let us calculate the term  $\int_{\mathbb{R}} X^* (\eta^+ H'_1 L\phi) dz_1$ . By formula (13) of Laplacian in the Fermi coordinate,

$$\begin{aligned} & \int_{\mathbb{R}} X^* (\eta^+ H'_1 L\phi) dz_1 \\ &= - \int_{\mathbb{R}} X^* (\eta^+ H'_1) \left[ A^{-1} \partial_{r_1}^2 + \frac{1}{2} \frac{\partial_{z_1} A}{A} \partial_{z_1} - \frac{1}{2} \frac{\partial_{r_1} A}{A^2} \partial_{r_1} \right] \phi^* dz_1 \\ &- \int_{\mathbb{R}} X^* (\eta^+ H'_1) r^{-1} [\partial_{r_1} \phi^* \partial_r r_1 + \partial_{z_1} \phi^* \partial_r z_1] dz_1 \\ &+ \int_{\mathbb{R}} [-X^* (\eta^+ H'_1) \partial_{z_1}^2 \phi^* + X^* (\eta^+ H'_1) (3\bar{u}^2 - 1) \phi] dz_1. \end{aligned}$$

We first estimate  $\int_{\mathbb{R}} X^* (\eta^+ H'_1) A^{-1} \partial_{r_1}^2 \phi^* dz_1$ . Differentiating the identity

$$\int_{\mathbb{R}} X^* (\eta^+ H'_1 \phi) dz_1 = 0 \quad (23)$$

with respect to  $r_1$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}} X^* (\eta^+ H'_1) \partial_{r_1} \phi^* dz_1 &= - \int_{\mathbb{R}} \partial_{r_1} [X^* (\eta^+ H'_1)] \phi^* dz_1 \\ &= - \int_{\mathbb{R}} \partial_{r_1} (\eta^{+*}) H' \phi^* dz_1 + \int_{\mathbb{R}} \eta^{+*} H'' h' \phi^* dz_1. \end{aligned}$$

Here  $H'$  and  $H''$  are evaluated at  $z_1 - h(r_1)$ . Therefore we could apply Lemma 10 and obtain

$$\begin{aligned} & \int_{\mathbb{R}} X^* (\eta^+ H'_1) \partial_{r_1} \phi^* \\ &= O \left( \|\phi^* (r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)} \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right). \end{aligned}$$

Similarly, differentiating (23) with respect to  $r_1$  twice, and using the fact that  $A^{-1} - \frac{1}{1+f'^2} = O(f'')$ , we get

$$\begin{aligned}
& \int_{\mathbb{R}} X^* (\eta^+ H_1') A^{-1} \partial_{r_1}^2 \phi^* dz_1 \\
&= \frac{1}{1+f'^2} \int_{\mathbb{R}} X^* (\eta^+ H_1') \partial_{r_1}^2 \phi^* dz_1 + O(\|\partial_{r_1}^2 \phi^*(r_1, \cdot)\|_{\infty} |f''(r_1)|) \\
&= -\frac{2}{1+f'^2} \int_{\mathbb{R}} \partial_{r_1} [X^* (\eta^+ H_1')] \partial_{r_1} \phi^* - \frac{1}{1+f'^2} \int_{\mathbb{R}} \partial_{r_1}^2 [X^* (\eta^+ H_1')] \phi^* \\
&+ O(\|\partial_{r_1}^2 \phi^*(r_1, \cdot)\|_{\infty} |f''(r_1)|) \\
&= O(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)}) + O(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}^2) \\
&+ O(\|\phi^*(r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)}) + O(\|\phi^*(r_1, \cdot)\|_{\infty} \|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}) \\
&+ O(\|\phi^*(r_1, \cdot)\|_{\infty} \|\partial_{r_1}^2 \phi^*(r_1, \cdot)\|_{\infty}).
\end{aligned}$$

Next, using  $\partial_{z_1} A = O(f'')$ , we infer

$$\int_{\mathbb{R}} \frac{\partial_{z_1} A}{A} \partial_{z_1} \phi^* X^* (\eta^+ H_1') dz_1 = O(f'' \|\partial_{z_1} \phi^*(r_1, \cdot)\|_{\infty}).$$

Observe that  $\partial_{r_1} A = O(|f''| + |f'''|)$ , thus

$$\int_{\mathbb{R}} \frac{\partial_{r_1} A}{A^2} \partial_{r_1} \phi^* X^* (\eta^+ H_1') dz_1 = O(|f''| \|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}) + O(|f'''| \|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}).$$

We also have

$$\begin{aligned}
& \int_{\mathbb{R}} r^{-1} X^* (\eta^+ H_1') [\partial_{r_1} \phi^* \partial_r r_1 + \partial_{z_1} \phi^* \partial_r z_1] \\
&= O(r^{-1} \|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}) + O(r^{-1} |f'| \|\partial_{z_1} \phi^*(r_1, \cdot)\|_{\infty}).
\end{aligned}$$

Another term we need to estimate is

$$\begin{aligned}
& \int_{\mathbb{R}} X^* (\eta^+ H_1') \partial_{z_1}^2 \phi^* + \int_{\mathbb{R}} X^* (\eta^+ H_1' (3\bar{u}^2 - 1) \phi) \\
&= \int_{\mathbb{R}} X^* (\eta^+ H_1') \partial_{z_1}^2 \phi^* + \int_{\mathbb{R}} X^* (\eta^+ H_1' (3H_1^2 - 1) \phi) \\
&+ 3 \int_{\mathbb{R}} X^* (\eta^+ H_1' (\bar{u}^2 - H_1^2) \phi).
\end{aligned}$$

To handle them, we integrate by parts for the first term, and for the last term we use the fact

$$\eta^+ (\bar{u}^2 - H_1^2) H_1' = O(\eta^+ (\mathcal{H}_1^s + 1) H_1') = O(e^{-D}).$$

We conclude that

$$\begin{aligned}
& \int_{\mathbb{R}} X^* (\eta^+ H_1') \partial_{z_1}^2 \phi^* + \int_{\mathbb{R}} X^* (\eta^+ H_1' (3\bar{u}^2 - 1) \phi) \\
&= O(\|\phi^*(r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f}) + O(\|\phi^*(r_1, \cdot)\|_{\infty} \|\partial_{r_1} \phi^*(r_1, \cdot)\|_{\infty}).
\end{aligned}$$

Finally, since  $P(\phi)$  is higher order term of  $\phi$ , the estimate of the term  $\int_{\mathbb{R}} X^*(\eta^+ H_1' P(\phi)) dz_1$  is trivial. Combining all these estimates, we get the desired result. ■

With the error term  $E(\bar{u})$  being understood, we would like to recall some properties of the operator  $L$ . The next result are essentially proved in [10], although here we need to do slight modifications due to the presence of boundary terms. For each  $r_1 \geq r_0$ , let

$$K_{r_1} := \left\{ (r, z) \in E, -f(r_1) + \frac{1}{f'(r_1)}(r - r_1) < z < f(r_1) - \frac{1}{f'(r_1)}(r - r_1) \right\},$$

and  $J_{r_1} := E \setminus K_{r_1}$ . Note that  $J_{r_0}$  is simply the set  $\hat{\mathbb{E}}$  introduced before.

**Lemma 17** *Suppose  $\varphi$  is a solution of the equation*

$$L\varphi = \Theta \text{ in } J_{r_1}.$$

*and  $\varphi(r, z) = \varphi(r, -z)$ ,  $\varphi_1^\parallel = 0$ . Then*

$$|\varphi^*(s, z)| \leq C \|\varphi\|_{L^\infty(\partial J_{r_1})} e^{r_1 - s} + \|\Theta\|_{L^\infty(J_{r_1})}, \text{ for each } s > r_1.$$

**Proof.** Let  $\theta$  be a cutoff function such that

$$\theta(Z) = \begin{cases} 1, & \text{if } Z \in J_{r_1} \text{ and } \text{dist}(Z, \partial J_{r_1}) > 1, \\ 0, & Z \notin J_{r_1}. \end{cases}$$

Consider the function  $\bar{\varphi} := \theta\varphi$ . Then  $\bar{\varphi}$  satisfies

$$L\bar{\varphi} = \theta\Theta + \Delta\theta\varphi + 2\nabla\theta\nabla\varphi. \quad (24)$$

Modify the function  $\bar{u}$  in  $E \setminus J_{r_1}$  if necessary, we could get a function, still denoted by  $\bar{u}$ , whose nodal curve are almost horizontal and around its nodal curve, it looks like the heteroclinic function  $H$ . With slight abuse of notation, the corresponding linearized operator will be still write as  $L$ . We could also assume that  $\bar{\varphi}$  satisfies the orthogonality condition.

Denote  $\Theta_1 = \theta\Theta$ ,  $\Theta_2 = \Delta\theta\varphi + 2\nabla\theta\nabla\varphi$ . For  $i = 1, 2$ , consider the equation

$$L\bar{\varphi}_i = \Theta_i + k_i(r_1)\eta^+ H_1' + (k_i\eta^+ H_1')^s \quad (25)$$

where  $\bar{\varphi}_i, \bar{k}_i$  are unknown functions and we require  $\int_{\mathbb{R}} X^*(\eta^+ H_i' \bar{\varphi}_i) dz_1 = 0, i = 1, 2$ . By the results of [10], one could find solution  $(\bar{\varphi}_i, \bar{k}_i)$  to this equation. This pair of solution is indeed unique and hence we have the following estimate for  $\bar{\varphi}_1$ :

$$\|\bar{\varphi}_1\|_{L^\infty} \leq C \|\Theta_1\|_{L^\infty}.$$

For the function  $\bar{\varphi}_2$ , since  $\Theta_2$  has better decay property than  $\Theta_1$ , we could estimate

$$|\bar{\varphi}_2(r, z)| \leq C \|\Theta_2\|_{L^\infty} e^{r_1 - r}.$$

Notice that

$$L(\bar{\varphi}_1 + \bar{\varphi}_2) = \Theta_1 + \Theta_2 + (k_1 + k_2)\eta^+ H'_1 + [(k_1 + k_2)\eta^+ H'_1]^s.$$

But the equation

$$L\bar{\varphi}_3 = \Theta_1 + \Theta_2 + k_3\eta^+ H'_1 + [k_3\eta^+ H'_1]^s$$

also has a unique pair of solution  $(\bar{\varphi}_3, k_3)$  if we require  $\int_{\mathbb{R}} X^*(\eta^+ H'_1 \bar{\varphi}_3) = 0$ . This implies that  $\bar{\varphi} = \bar{\varphi}_1 + \bar{\varphi}_2$ , the conclusion of this proposition readily follows.  $\blacksquare$

Once we have an estimate in  $L^\infty$  norm, we could get a corresponding Holder norm estimate by Schauder's elliptic estimate. Now we proceed to the proof of Proposition 11.

**Proof of Proposition 11.** Recall that

$$\begin{aligned} & \int_{\mathbb{R}} X^*(\eta^+ H'_1 E(\bar{u})) \\ &= \int_{\mathbb{R}} X^*(\eta^+ H'_1 E(\mathcal{H}_1)) + \int_{\mathbb{R}} X^*(\eta^+ H'_1 3(\mathcal{H}_1^s + 1)(\mathcal{H}_1^2 - 1)) \\ &+ \int_{\mathbb{R}} X^*(\eta^+ H'_1 E(\mathcal{H}_1^s)) + \int_{\mathbb{R}} X^*(\eta^+ H'_1 3(\mathcal{H}_1 + 1)(\mathcal{H}_1^s + 1)^2). \end{aligned}$$

We have already analyzed the first two terms. For the third term, slightly abuse the notation, we have

$$\begin{aligned} \int_{\mathbb{R}} X^*(\eta^+ H'_1 E(\mathcal{H}_1^s)) dz_1 &= \int_{\mathbb{R}} X^*(\eta^+ H'_1) A^{-1} (-h'' H' + h'^2 H'') dz_1 \\ &+ \int_{\mathbb{R}} X^*(\eta^+ H'_1) \left( \frac{\partial_{z_2} A}{2A} + r^{-1} \partial_r z_2 \right) H' dz_1 \\ &+ \int_{\mathbb{R}} X^*(\eta^+ H'_1) \left( \frac{\partial_{r_2} A}{2A^2} - r^{-1} \partial_r r_2 \right) (h' H') dz_1. \end{aligned}$$

It should be pointed out that here  $A, h$  are evaluated at  $r_2$  and  $H'$  is evaluated at  $z_2 - h(r_2)$ , and  $h'$  means the derivative with respect to  $r_2((r_2, z_2))$  being the Fermi coordinate with respect to  $z = -f(r)$ , rather than  $r_1$ . This indeed makes the analysis a little more complicated. We have

$$\begin{aligned} & \int_{\mathbb{R}} X^*(\eta^+ H'_1) A^{-1} (-h'' H' + h'^2 H'') dz_1 \\ &+ \int_{\mathbb{R}} X^*(\eta^+ H'_1) \left( \frac{\partial_{r_2} A}{2A^2} - r^{-1} \partial_r r_2 \right) (h' H') dz_1 \\ &= o(e^{-\sqrt{2}D}). \end{aligned}$$

For the same reason,

$$\int_{\mathbb{R}} X^*(\eta^+ H'_1) \left( \frac{\partial_{z_2} A}{2A} + r^{-1} \partial_r z_2 \right) H' dz_1 = o(e^{-D}).$$

It follows that

$$\int_{\mathbb{R}} X^* (\eta^+ H_1' E (\mathcal{H}_1^s)) dz_1 = o \left( e^{-\sqrt{2}D} \right).$$

Similarly,

$$\int_{\mathbb{R}} X^* \left( \eta^+ H_1' 3 (\mathcal{H}_1 + 1) (\mathcal{H}_1^s + 1)^2 \right) = o \left( e^{-\sqrt{2}D} \right).$$

It then follows from Lemma 12 and Lemma 14 that

$$\begin{aligned} & \int_{\mathbb{R}} X^* (\eta^+ H_1' E (\bar{u})) \\ &= - (1 + o(1)) \left[ \frac{\mathbf{c}_0 h''}{1 + f'^2} + \frac{\mathbf{c}_0 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 f'}{r_1 \sqrt{1 + f'^2}} \right] \\ & \quad - \mathbf{c}_1 (1 + o(1)) e^{-\sqrt{2}D(r_1)} + O(h'^2) + O(h' f'') \\ & \quad + O(h' f''') + O(h' r^{-1}) + O(h'' f''). \end{aligned}$$

This together with Lemma 16 leads to the following equation:

$$\begin{aligned} & \frac{\mathbf{c}_0 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 h''}{1 + f'^2} + \frac{\mathbf{c}_0 f'}{r_1 \sqrt{1 + f'^2}} - (1 + o(1)) \mathbf{c}_1 e^{-\sqrt{2}D} \\ &= O(h'' f'') + O(h'^2) + O(h' f'') + O(h' r^{-1}) + O(h' f''') \\ & \quad + O \left( \|\phi^* (r_1, \cdot)\|_{\infty}^2 \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)} \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} \right) \\ & \quad + O \left( \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty} e^{-\sqrt{2}f(r_1)} \right) + O \left( \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty}^2 \right) + O \left( \|\phi^* (r_1, \cdot)\|_{\infty} \|\partial_{r_1}^2 \phi^* (r_1, \cdot)\|_{\infty} \right) \\ & \quad + O(f'' \|\partial_{z_1} \phi^* (r_1, \cdot)\|_{\infty}) + O(f'' \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty}) + O(f''' \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty}) \\ & \quad + O(r^{-1} \|\partial_{r_1} \phi^* (r_1, \cdot)\|_{\infty}) + O(r^{-1} f' \|\partial_{z_1} \phi^* (r_1, \cdot)\|_{\infty}). \end{aligned} \tag{26}$$

On the other hand,

$$L\phi = [E(\bar{u})]^{\perp} + [E(\bar{u})]^{\parallel} + P(\phi).$$

By Lemma 16,  $[E(\bar{u})]^{\parallel} + P(\phi)$  is small compared to  $\phi$ . In view of Lemma 17, we need to estimate  $[E(\bar{u})]^{\perp}$ .

By (19),

$$\begin{aligned} [E(\bar{u})]^{\perp} &= [E(\mathcal{H}_1)]^{\perp} + [3(\mathcal{H}_1^s + 1)(\mathcal{H}_1^2 - 1)]^{\perp} \\ & \quad + [E(\mathcal{H}_1^s)]^{\perp} + [3(\mathcal{H}_1 + 1)(\mathcal{H}_1^s + 1)^2]^{\perp}. \end{aligned}$$

Let us analyze the term  $[E(\mathcal{H}_1^s)]^{\perp}$ . We know that

$$\begin{aligned} E(\mathcal{H}_1^s) &= A^{-1} (-h'' H' + h' H'') + \left( \frac{\partial_{z_2} A}{2A} + r^{-1} \partial_r z_2 \right) H' \\ & \quad + \left( \frac{\partial_{r_2} A}{2A^2} - r^{-1} \partial_r r_2 \right) h' H'. \end{aligned}$$

Hence in the upper plane, due to the presence of  $H'$  in each term, we find that directly that

$$[E(\mathcal{H}_1^s)]^\perp = O\left(e^{-\sqrt{2}f}\right). \quad (27)$$

Note that this is not an optimal estimate and each term in  $E(\mathcal{H}_1^s)$  is multiplied by  $h', h'', f''$  or  $r^{-1}f'$ . However, these terms are in general not evaluated at  $r_1$ . One also has

$$\left[3(\mathcal{H}_1 + 1)(\mathcal{H}_1^s + 1)^2\right]^\perp = o\left(e^{-\sqrt{2}D}\right).$$

The above estimates combined with Lemma 13 and 14 leads to

$$\begin{aligned} [E(\bar{u})]^\perp &= [E(\mathcal{H}_1^s)]^\perp + O(f''^2) + O(r^{-2}f'^2) + O(h''f'') + O(h'r^{-1}) \\ &\quad + O(h'^2) + O(h'^2) + O(h'f'') + O(h'f''') + O(e^{-\sqrt{2}D}). \end{aligned} \quad (28)$$

Notice that there is a term  $f''^2$  appeared in the right hand side. Insert (26) into (28) we get

$$\begin{aligned} |(E(\bar{u}))^\perp| &= [E(\mathcal{H}_1^s)]^\perp + O(r^{-2}f'^2) + O(h'r^{-1}) \\ &\quad + O(h'^2) + O(h'f''') + O(e^{-\sqrt{2}D}) \\ &\quad + O(h'h'') + O(hf''') + O(h''^2) \\ &\quad + O\left(\|\phi^*(r_1, \cdot)\|_\infty^2\right) + O\left(\|\partial_{r_1}\phi^*(r_1, \cdot)\|_\infty^2\right) \\ &\quad + O\left(\|\partial_{z_1}\phi^*(r_1, \cdot)\|_\infty^2\right). \end{aligned} \quad (29)$$

Note that the term involving  $h$  and its derivatives are essentially controlled by  $\phi$ . Using similar arguments as that of Lemma 17, we find that for  $s > r_1$ ,

$$|\phi^*(s, z_1)| \leq C\|e^{-D}\|_{L^\infty((r_1, +\infty))} + C\|r^{-2}f'^2\|_{L^\infty((r_1, +\infty))} + \|\phi\|_{L^\infty(\partial J_{r_1})} e^{r_1-s}. \quad (30)$$

We remark that there is the term  $[E(\mathcal{H}_1^s)]^\perp$  appearing in (29). Initially, by (27), it is only of the order  $O(e^{-f})$ . But we would using a bootstrap argument to show that this term is indeed of the order  $O(e^{-D})$ .

The estimate (30) could be applied repeatedly. This yields that for some large but fixed constant  $l$ ,

$$\begin{aligned} \phi(r, z) &\leq Ce^{-D(r-l)} + \frac{C}{(r-l)^2} + C\|\phi\|_{L^\infty(\partial J_{r-l})} e^{-l} \\ &\leq Ce^{-D(r-l)} + \frac{C}{(r-l)^2} \\ &\quad + C^2 \left[ e^{-D(r-2l)} + \frac{1}{(r-2l)^2} + \|\phi\|_{L^\infty(\partial J_{r-2l})} e^{-l} \right] e^{-l} \\ &\leq Ce^{-D(r)} + \frac{C}{r^2} + C\|\phi\|_{L^\infty(\partial J_{r_0})} e^{-r}. \end{aligned}$$



The last inequality follows from

$$\sum_{j=1}^k \frac{1}{(r-jl)^2} e^{-jl} \leq \frac{C}{r^2},$$

and by Lemma 6,  $f'$  is small, which together with  $D = (2 + o(1))f$  implies that

$$\sum_{j=1}^k e^{-D(r-jl)} e^{-jl} \leq C e^{-D(r)}.$$

This concludes the proof. ■

Up to now, we have assumed (A). But a priori, we don't know whether this is fulfilled or not. In the sequel, we sketch the proof without this assumption. The basic principle is, using equation (26) and similar arguments above, one could prove that assumption (A) indeed holds.

**Proof of Proposition 11 without Assumption (A).** The main reason that we introduce the Assumption (A) is that in the the error  $E(\bar{u})$  is related to  $e^{-2\sqrt{2}\bar{d}}$  and we could control it by  $e^{-D}$ . Now without Assumption (A), to handle the term  $O(e^{-2\sqrt{2}\bar{d}})$ , we would like to use the following basic geometrical fact concerning the Fermi coordinate (see [19] for a proof): Using the fact that  $f'(r_1)$  is small, there exists a point  $\bar{r}$  ( $\bar{r}$  may be equal to  $r_1$ ) with

$$|\bar{r} - r_1| = o(\min\{d_1(r_1), d_2(r_1)\})$$

such that

$$\min\{d_1(r_1), d_2(r_1)\} \geq \frac{1}{|f''(\bar{r})|}. \quad (31)$$

Let us still assume  $|\bar{d}'| \leq \frac{1}{2}$  and  $|\bar{d}''| \leq C$  for the moment. Then repeating the previous arguments, one could show that for  $r_1 > r_0$ ,

$$|f''(r_1)| \leq C e^{-r_1} + \frac{C}{r_1} + C e^{-\sqrt{2}D(r_1)} + C \left\| e^{-2\sqrt{2}\bar{d}} \right\|_{L^\infty(r_1, +\infty)}. \quad (32)$$

We claim there exists a universal constant  $C_0$  such that

$$\bar{d}(r_1) = 3f(r_1) \text{ for } r_1 > C_0. \quad (33)$$

Suppose to the contrary that (33) is not satisfied at some large point  $s$ . Without loss of generality we assume  $\left\| e^{-2\sqrt{2}\bar{d}} \right\|_{L^\infty((s, +\infty))} = e^{-2\sqrt{2}\bar{d}(s)}$ . By (31), we could find  $s_1$  with

$$|s_1 - s| = o(\min\{d_1(s), d_2(s)\}) = o(f(s))$$

such that

$$\min\{d_1(s), d_2(s)\} > \frac{1}{|f''(s_1)|}. \quad (34)$$

We would like to show that for this point  $s_1$ , necessarily

$$\frac{1}{|f''(s_1)|} \geq 3f(s). \quad (35)$$

Once this is proved, we then get a contradiction and hence the proof will be accomplished.

To prove (35), we could assume  $s_1 \neq s$ , otherwise using estimate (32), we obtain (35). For the point  $s_1$ , we have

$$\begin{aligned} |f''(s_1)| &\leq Ce^{-s_1} + \frac{C}{s_1} + Ce^{-\sqrt{2}D(s_1)} + C \left\| e^{-2\sqrt{2}d} \right\|_{L^\infty(s_1, +\infty)} \\ &= Ce^{-s_1} + \frac{C}{s_1} + Ce^{-\sqrt{2}D(s_1)} + Ce^{-2\sqrt{2}\bar{d}(\bar{s}_1)} \end{aligned}$$

for some point  $\bar{s}_1 \geq s_1$ . If  $Ce^{-2\sqrt{2}\bar{d}(\bar{s}_1)} < \frac{1}{2}|f''(s_1)|$ , then

$$|f''(s_1)| \leq Ce^{-s_1} + \frac{C}{s_1} + Ce^{-\sqrt{2}D(s_1)},$$

which implies (35). Hence we could assume

$$Ce^{-2\sqrt{2}\bar{d}(\bar{s}_1)} \geq \frac{1}{2}|f''(s_1)|. \quad (36)$$

This in particular implies that

$$\bar{d}(\bar{s}_1) \leq \frac{1}{2\sqrt{2}} \left( \ln \frac{2C}{f''(s_1)} \right) \leq \ln f(s).$$

For the point  $\bar{s}_1$ , by (31), we could find a point  $s_2$  with

$$s_2 \geq \bar{s}_1 + o(\min\{d_1(\bar{s}_1), d_2(\bar{s}_1)\}) \geq s_1 - \ln f(s),$$

such that

$$\min\{d_1(\bar{s}_1), d_2(\bar{s}_1)\} \geq \frac{1}{f''(s_2)}.$$

Note that by (36),

$$Ce^{-\frac{2\sqrt{2}}{f''(s_2)}} \geq \frac{1}{2}|f''(s_1)|.$$

Now repeat this argument and we get a sequence  $\{s_n\}$  with

$$s_{n+1} \geq s_n - \ln \frac{1}{|f''_n(s)|}, \quad (37)$$

such that

$$Ce^{-\frac{2\sqrt{2}}{f''(s_{n+1})}} \geq \frac{1}{2}|f''(s_n)|.$$

If the process stops at the step  $n_0 < f(s)$ . Then

$$f''(s_{n_0}) \leq Ce^{-s_{n_0}} + \frac{1}{s_{n_0}} + e^{-D(s_{n_0})}.$$

By (37),  $s_{n_0} > s - f(s)$  and  $D(s_{n_0}) > f(s)$ . This then implies (35). On the other hand, if the process doesn't stop at the step  $n_0 > f(s)$ , then the sequence  $\rho_n := \frac{2\sqrt{2}}{|f''(s_n)|}$  satisfies

$$\rho_n \geq Ce^{\rho_{n+1}}. \quad (38)$$

(38) leads to the inequality

$$\rho_1 \geq Cn_0^2 \geq Cf^2(s),$$

This also implies (35).

Let us return back to the assumption that  $|\bar{d}'| \leq \frac{1}{2}$  and  $|\bar{d}''| \leq C$ . Essentially, this assumption is to make sure that the cutoff function  $\eta$  has bounded first and second derivatives and thus the error of approximate solution could be uniformly controlled. However, with the original definition of  $\bar{d}$  this assumption may not be true. To overcome this difficulty, we should modify the function  $\bar{d}$  (that is, one needs to modify the domain  $\mathcal{B}_u$ ). Let us be more precise. Fix a small positive constant  $\delta$ . We define new function  $\hat{d}$  by

$$\hat{d}(r_1) := \inf \{ \bar{d}(s) + \delta(r_1 - s) : r_0 < s < r_1; \bar{d}(s) - \delta(r_1 - s) : s > r_1 \}.$$

Modify  $\hat{d}$  such that it becomes  $C^2$ . We then define the domain  $\mathcal{B}_u$  using this new function  $\hat{d}$ . Using similar arguments as before, we could show (33).

Now (33) implies that the size of the Fermi coordinate is actually large enough for our purpose, that is, Assumption (A) is indeed satisfied. ■

With all these preparation, we proceed to prove Proposition 4, the main result of this section.

**Proof of Proposition 4 and Theorem 2.** We split the proof into several steps.

*Step 1.* We first show that there exists a universal constant  $C, \delta > 0$  such that

$$p(r) \geq C + \left( \frac{\sqrt{2}}{2} + \delta \right) \ln r.$$

We would like to use equation (26). Note that there is a term involving the third derivative of  $f$  in this equation, although one expects that  $f'''$  decays like  $r^{-3}$ , a priori we don't have any decay information for it. However, we at least know that  $f'''(r_1)$  tends to zero as  $r_1$  tending to infinity. Our aim is to show the following estimate(not optimal):

$$|f'''(r_1)| \leq Cr_1^{-1} + Ce^{-D(r_1)}. \quad (39)$$

To obtain this estimate, we would like to differentiate equation (26) with respect to  $r_1$ . Using the fact that  $f^{(3)}$  and  $f^{(4)}$  are small for  $r_1$  large(at this stage, we

don't know the decay rate for these two terms, but they are multiplied by terms related to  $\phi$ , we obtain the following estimate (which is not optimal but enough for our purpose)

$$\begin{aligned}
& f''' + \sqrt{1 + f'^2} h''' \\
&= O(h'') + O(r_1^{-1}) + O(e^{-\sqrt{2}D}) \\
&+ O(h') + O(\|\phi^*(r_1, \cdot)\|_\infty) + O(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty e^{-\sqrt{2}f(r_1)}) \\
&+ O(\|\partial_{r_1}^2 \phi^*(r_1, \cdot)\|_\infty) + O(\|\partial_{r_1} \partial_{z_1} \phi^*(r_1, \cdot)\|_\infty).
\end{aligned}$$

Hence by the  $C^{2,\alpha}$  estimate of  $\phi$ , to prove (39), it will be suffice to obtain an estimate for the function  $h'''$ , which is essentially controlled by  $\partial_{r_1}^3 \phi^*$ . To achieve this, we consider the equation

$$L\phi = [E(\bar{u})]^\perp + [E(\bar{u})]^\parallel + P(\phi). \quad (40)$$

Here we have in mind that by Lemma 16,  $[E(\bar{u})]^\parallel$  is expressed as a small order term of  $\phi$ . We also have the estimate (29) for the source term  $[E(\bar{u})]^\perp$ . Now differentiate equation (40) with respect to  $r_1$ , use the  $L^\infty$  norm estimate of  $\phi$  and the fact that

$$\partial_{r_1} [E(\bar{u})]^\perp = O(r_1^{-1}) + O(e^{-D}) + o(h'''),$$

we find that for  $r_1$  large,

$$|\partial_{r_1}^3 \phi^*| \leq Cr_1^{-1} + Ce^{-D}.$$

This then implies (39).

With the decay estimate of  $f'''$  available, equation (26) could be refined to

$$\left( \frac{r_1 p'}{\sqrt{1 + p'^2}} \right)' = \frac{c_1}{c_0} r_1 (1 + o(1)) e^{-\sqrt{2}D} + O(r_1^{-2}) := I_1 + I_2. \quad (41)$$

Here  $p = f + \sqrt{1 + f'^2} h$  is the function introduced in Lemma 12. At this stage, one of the technical difficulty is that this equation involves the function  $D$ . As we mentioned before, we expect that  $D$  is very close to  $2p$ . But without a priori estimate for  $f'$  and  $f$ ,  $|D - 2p|$  in principle could be large. In any case, we at least know that  $D < 2p$ .

To proceed, we shall fix a small constant  $\bar{\delta} > 0$  which will be determined later. For any interval  $(t_1, t_2) \subset (r_0, +\infty)$ , integrating equation (41) in  $r_1$  yields

$$\frac{t_2 p'(t_2)}{\sqrt{1 + p'^2(t_2)}} - \frac{t_1 p'(t_1)}{\sqrt{1 + p'^2(t_1)}} = \int_{t_1}^{t_2} I_1 + \int_{t_1}^{t_2} I_2. \quad (42)$$

Keep in mind that  $\int_{t_1}^{t_2} I_1$  is always positive, and by  $|I_2(r)| \leq C_0 r^{-2}$ ,

$$\left| \int_{t_1}^{t_2} I_2 \right| \leq \frac{C_0}{t_1}. \quad (43)$$

Set  $t^* := \max\{\frac{2C_0}{\delta}, r_0\}$ . We consider two cases. Case 1:

$$\frac{t^* p'(t^*)}{\sqrt{1 + p'^2(t^*)}} \geq \frac{\sqrt{2}}{2} + \bar{\delta}.$$

Then by (42) and (43), for all  $t > t^*$ ,

$$\frac{tp'(t)}{\sqrt{1 + p'^2(t)}} \geq \frac{\sqrt{2}}{2} + \frac{\bar{\delta}}{2}.$$

Case 2:

$$\frac{t^* p'(t^*)}{\sqrt{1 + p'^2(t^*)}} < \frac{\sqrt{2}}{2} + \bar{\delta}.$$

In this case, let  $(t^*, \hat{t})$  be an interval such that

$$\frac{tp'(t)}{\sqrt{1 + p'^2(t)}} \leq \frac{\sqrt{2}}{2} + 2\bar{\delta}, \forall t \in (t^*, \hat{t}).$$

Then the fact that  $|p'| = o(1)$  and a simple integration yield

$$p(t) \leq \left(\frac{\sqrt{2}}{2} + 3\bar{\delta}\right) \ln t + C, \forall t \in (t^*, \hat{t}).$$

This in particular implies that  $D(r) - 2f(r) = o(1)$  and hence for any  $t_1, t_2 \in (t^*, \hat{t})$ ,

$$\int_{t_1}^{t_2} s e^{-\sqrt{2}D(s)} ds \geq \frac{C}{t_1^{6\sqrt{2}\bar{\delta}}} - \frac{C}{t_2^{6\sqrt{2}\bar{\delta}}}.$$

We choose  $\bar{\delta}$  such that  $6\sqrt{2}\bar{\delta} < 1$ . Then by inequality (43), if  $t_2$  is large,

$$\int_{t_1}^{t_2} I_1 + \int_{t_1}^{t_2} I_2 > 0.$$

This implies

$$\frac{t_2 p'(t_2)}{\sqrt{1 + p'^2(t_2)}} > \frac{t_1 p'(t_1)}{\sqrt{1 + p'^2(t_1)}}.$$

Combing the analysis for Case 1 and Case 2, we find that there exist universal constants  $\delta$  and  $C_1$  such that

$$rp'(r) > \frac{\sqrt{2}}{2} + \delta, \quad r > C_1. \quad (44)$$

This proves Theorem 2.

With the lower bound (44) available, one could show that

$$e^{-\sqrt{2}D} = e^{-2\sqrt{2}p} + O\left(r^{-(2+\alpha)}\right).$$

It also follows that

$$p'(r) = \frac{k}{r} + O(r^{-1-\alpha}),$$

for some  $\alpha > 0$ . Here  $k$  is the growth rate of  $u$ . Therefore,

$$p(r) = k \ln r + O(r^{-\alpha}) + C.$$

With this estimate at hand, we find that  $u_n$  converges to a two-end solution  $u_0$  strongly. ■

### 3 Moduli space theory of two-end solutions

#### 3.1 Preliminary results

Generally speaking, the structure of the set of bounded entire solutions to the Allen-Cahn equation could potentially be very complicated. However, if one impose certain conditions at infinity for the solution, then it could be simpler. The moduli space theory for multiple-end solutions of the Allen-Cahn equation in  $\mathbb{R}^2$  was developed in [9]. This theory tells us that if  $u$  is a  $2k$ -end solutions in  $\mathbb{R}^2$  and nondegenerate, then around  $u$  (in suitable sense), the set of  $2k$ -end solutions is actually a  $2k$ -dimensional manifold. This fact has been used in an essential way in the classification of four-end solutions of Allen-Cahn equation in  $\mathbb{R}^2$  ([19]). In this section, we would like to develop the corresponding moduli space theory for two-end solutions in  $\mathbb{R}^3$ . Our main result states that the moduli space of two-end solutions has a structure of real analytic variety of formal dimension 1.

To begin with, let us recall some preliminary results about real analytic operators. Compared to  $C^\infty$  operators, real analytic operators has better structures. We refer to [5], [6], [7] for more details.

Let  $X$  and  $Y$  be Banach spaces and  $U$  an open subset of  $X$ . We first recall the notion of real analytic operator.

**Definition 18** *A map  $F : U \rightarrow Y$  is real analytic at  $x_0 \in U$  if there exists a  $\delta > 0$  such that*

$$F(x) - F(x_0) = \sum_{k=1}^{+\infty} m_k((x - x_0), \dots, (x - x_0)), \text{ for } |x - x_0| < \delta,$$

where  $m_k$  is a symmetric  $k$ -linear operator, and there exists  $r > 0$  such that

$$\sup_{k \geq 0} r^k \|m_k\| < +\infty.$$

The function is said to be real analytic on  $U$  if it is real analytic at every point of  $U$ .

Let  $F : U \rightarrow Y$  be a real analytic functional. Suppose that  $dF(x)$  is a Fredholm operator of index 1. Assume there exists a map  $\Lambda : (0, \varepsilon) \rightarrow Y$  such that  $F(\Lambda(s)) = 0$ , and  $dF(\Lambda(s)) : X \rightarrow Y$ , is surjective for all  $s \in (0, \varepsilon)$ . Let

$$S = \{x \in U : F(x) = 0\}.$$

The following theorem has been proved in the book of Buffoni and Toland[5].

**Theorem 19** *Suppose all bounded closed subsets of  $S$  are compact in  $X$ . Then there exists an extension of  $\Lambda$ , denoted by  $\bar{\Lambda}$  :*

$$(0, +\infty) \rightarrow X,$$

*satisfying: (1)  $\bar{\Lambda}$  is continuous. (2) The set of points where  $dF$  is not surjective has no accumulation points. (3) One of the following happens: (i)  $\|\bar{\Lambda}(s)\| \rightarrow +\infty$ , as  $s \rightarrow +\infty$ ; (ii)  $\bar{\Lambda}(s)$  approaches the boundary of  $U$  as  $s$  tends to infinity; (iii)  $\bar{\Lambda}((0, +\infty))$  is a closed loop.*

Basically, this theorem tells us that if one has a real analytic variety which comes from the zero set of a real analytic operator and its formal dimension is one, and assume further that on the variety there are some points where the operator is surjective, then under certain compactness assumption, one could find a continuous path of solutions where at most finitely many solutions are degenerate (the linearized operator is not surjective).

### 3.2 The real analytic structure of the moduli space

Let  $u$  be a two-end solution. Then  $u$  satisfies (2) and there are constants  $k$  and  $c_k \in \mathbb{R}$  such that

$$\|u(r, \cdot) - H(\cdot - k \ln r - c_k)\|_{L^\infty([0, +\infty))} \rightarrow 0, \text{ as } r \rightarrow +\infty. \quad (45)$$

We also assume  $k > \sqrt{2}$ . The moduli space theory of noncompact geometric objects with controlled geometry at infinity (minimal surfaces with finite total curvature, singular Yamabe metrics, constant mean curvature surfaces with De-launay ends) has been developed in [21], [23], [25]. In this paper, we will not investigate the general moduli space theory for the Allen-Cahn equation in dimension three. Instead, we shall only consider those solutions satisfying (2). Our aim is to show that the set of solutions of (2) satisfying (45) has the structure of a real analytic variety with formal dimension 1. Additionally, if a solution  $u$  is nondegenerate, then around  $u$ , this real analytic variety is actually a one dimensional real analytic manifold.

Consider the function

$$f_{k,b}(r) = k \cosh^{-1}(k^{-1}r) + b.$$

where  $k$  and  $b$  are real parameters,  $k > \sqrt{2}$ . Obviously,  $f$  represents a vertically translated catenoidal end and we have in mind that  $f$  is the asymptotic

curve of the nodal line of a solution  $u$ . The moduli space theory for two-end solutions will, roughly speaking, state that around a given solution  $u$  there is a one dimensional family of solutions, with  $k$  or  $b$  being possible candidates for the local parameters. To make this statement more precise, we need to analyze the mapping property of the linearized Allen-Cahn operator around  $u$ .

Let us introduce some notations. As in Section 2, we shall also carry out the analysis through the Fermi coordinate. Let  $\mathcal{B}_u$  be the domain where the Fermi coordinate of  $z = f(r)$  is well defined (suitably modified if necessary). We still use  $\eta$  to denote a smooth cutoff function supported in  $\mathcal{B}_u$ . Similarly, we have the cutoff function  $\eta^+$  supported in  $\mathcal{B}_u \cap \mathbb{E}^+$ . Similarly, we have the map  $X : (r_1, z_1) \rightarrow (r, z)$ .

We need to work in suitable functional spaces  $S_1, S_2$  which will be described now. Let  $C_1 > 0$  be a fixed constant and  $\delta > 0$  small. A  $C^{2,\alpha}$  function  $\Xi \in S_1$ , if and only if  $\Xi(r, z) = \Xi(r, -z)$  and it satisfies the following condition: In  $\mathbb{E}^+$ , when  $r_1 > C_1$ ,

$$X^* \Xi(r_1, z_1) = p(r_1) X^* \eta^+(r_1, z_1) H'(z_1) + X^* \psi(r_1, z_1),$$

where  $\int_{\mathbb{R}} X^* (\eta^+ \psi) H' dz_1 = 0$ ,

$$\left\| (1+r_1)^2 p \right\|_{L^\infty} + \left\| (1+r_1)^3 p' \right\|_{L^\infty} + \left\| (1+r_1)^4 p'' \right\|_{C^{0,\alpha}} < +\infty,$$

and

$$\left\| (1+r_1)^4 X^* \psi \right\|_{C^{2,\alpha}} < +\infty.$$

Similarly, a  $C^{0,\alpha}$  function  $\Xi \in S_2$  if and only if  $\Xi(r, z) = \Xi(r, -z)$  and it satisfies the following condition: In  $\mathbb{E}^+$ , when  $r_1 > C_1$ ,

$$X^* \Xi(r_1, z_1) = p(r_1) X^* \eta^+(r_1, z_1) H'(z_1) + X^* \psi(r_1, z_1),$$

where  $\int_{\mathbb{R}} X^* (\eta^+ \psi) H' dz_1 = 0$ ,

$$\left\| (1+r_1)^4 p \right\|_{C^{0,\alpha}} < +\infty.$$

and

$$\left\| (1+r_1)^4 X^* \psi \right\|_{C^{0,\alpha}} < +\infty.$$

Let  $\mathcal{L}$  be the linearized operator of the Allen-Cahn equation around a two-end solution  $u$ , that is,

$$\mathcal{L} = \mathcal{L}_u := \Delta_{(r,z)} + r^{-1} \partial_r + 1 - 3u^2.$$

For each point  $p$ , let  $(r_1(p), z_1(p))$  be its Fermi coordinate with respect to  $f_{k,b}$ . Let  $p'_{s,t}$  be the point whose Fermi coordinate with respect to the curve  $z = f_{k+s,b+t}$  is still equal to  $(r_1(p), z_1(p))$ . For  $p$  in the upper half plane, now let  $\Phi_{s,t}$  be the map defined by

$$\Phi_{s,t}(p) = \eta^+ p'_{s,t} + (1 - \eta^+) p,$$



while for  $p = (r, z)$  in the lower half plane, we let

$$\Phi_{s,t}(p) = -\Phi_{s,t}(r, -z).$$

In this way, we have defined the family of maps  $\Phi_{s,t}$  which is even with respect to the  $r$  axis. For  $|s|, |t|$  small,  $\Phi_{s,t}$  is a diffeomorphism. By definition  $\Phi_{0,0}(p) = p$ . Similarly, we introduce the family of diffeomorphism

$$\Psi_{s,t}(p) := \eta p'_{s,t} + (1 - \eta)p.$$

Similar arguments as that of Section 4 shows that at far away an axially symmetric two-end solution  $u$  could be written as:

$$u = \mathcal{H}_1 + \mathcal{H}_1^s + 1 + \phi.$$

Here

$$\mathcal{H}_1 = \eta H_1 + (1 - \eta) \frac{H_1}{|H_1|},$$

where

$$X^* H_1(r_1, z_1) = H(z_1 - h(r_1))$$

and

$$\int_{\mathbb{R}} X^*(\eta^+ \phi) H' dz_1 = 0.$$

Notice that  $\phi$  decays like  $r^{-4}$  at infinity.

For  $(s, t, \psi) \in \mathbb{R}^2 \oplus S_1$ , we then define a family of function  $u_{s,t,\psi}$  such that at far away

$$u_{s,t,\psi} = \mathcal{H}_1 \circ \Psi_{s,t}^{-1} + [\mathcal{H}_1 \circ \Psi_{s,t}^{-1}]^s + (\phi + \psi) \circ \Phi_{s,t}^{-1},$$

and in a fixed large ball,  $u_{s,t,\psi} = u$ . Clearly, for  $u_{0,0,0} = u$ . Consider the non-linear map  $N : \mathbb{R}^2 \oplus S_1 \rightarrow S_2$  given by

$$(s, t, \psi) \rightarrow [\Delta u_{s,t,\psi} + u_{s,t,\psi} - u_{s,t,\psi}^3] \circ \Phi_{s,t}.$$

The reason that  $N$  maps  $\mathbb{R}^2 \oplus S_1$  into  $S_2$  lies in the asymptotic behavior of  $u$ , see Section 2. We have

$$\begin{aligned} \partial_s N(s, t, \psi) &= [\mathcal{L}_{u_{s,t,\psi}} \partial_s u_{s,t,\psi}] \circ \Phi_{s,t} + \partial_s \Phi_{s,t} \cdot \nabla [\Delta u_{s,t,\psi} + u_{s,t,\psi} - u_{s,t,\psi}^3] \circ \Phi_{s,t}, \\ \partial_t N(s, t, \psi) &= [\mathcal{L}_{u_{s,t,\psi}} \partial_t u_{s,t,\psi}] \circ \Phi_{s,t} + \partial_t \Phi_{s,t} \cdot \nabla [\Delta u_{s,t,\psi} + u_{s,t,\psi} - u_{s,t,\psi}^3] \circ \Phi_{s,t}, \\ \partial_\psi N(s, t, \psi) G &= [\mathcal{L}_{u_{s,t,\psi}} G] \circ \Phi_{s,t}. \end{aligned}$$

In particular, although  $DN$  is not exactly equal to  $\mathcal{L}$ , it is a small perturbation of it for  $s, t$  small. Observe that when  $(s, t, \psi) = (0, 0, 0)$ ,  $DN$  actually is equal to  $\mathcal{L}$ . Let

$$\begin{aligned} \gamma_1 &= \partial_s u_{s,t,\psi}|_{(s,t,\psi)=(0,0,0)}, \\ \gamma_2 &= \partial_t u_{s,t,\psi}|_{(s,t,\psi)=(0,0,0)}. \end{aligned}$$

Introduce the deficiency space

$$\mathcal{D} = \text{span} \{ \gamma_1, \gamma_2 \}.$$

The next result concerns the mapping property of the linearized operator  $\mathcal{L}$  and is the main result of this section.

**Proposition 20** *The operator  $\mathcal{L} : S_1 \oplus \mathcal{D} \rightarrow S_2$  is a Fredholm operator of index 1.*

**Proof.** Let us prove the result under the additional assumption that  $f(0)$  is large and  $\|f'\|_{L^\infty}, \|f''\|_{L^\infty}$  are very small. In the general case, we could modify the function  $u$  inside a compact set into a function whose nodal lines are almost parallel, and use the fact that the corresponding linearized operator is a compact perturbation (thus the Fredholm index is preserved) of  $\mathcal{L}$ . We shall split the proof into several steps.

**Step 1.** For each function  $\Theta \in S_2$ , we shall find a solution  $\Xi$  to the equation

$$\mathcal{L}\Xi = \Theta. \quad (46)$$

By the definition of  $S_2$ , in  $\mathbb{E}^+$ , for  $r_1$  large,

$$X^*\Theta(r_1, z_1) = \omega(r_1) X^*\eta^+ H'(z_1) + X^*\varphi(r_1, z_1),$$

for some function  $\omega$  and  $\varphi$ , where  $\int_{\mathbb{R}} X^*(\eta^+ \varphi) H' = 0$ . To solve (46), adopt the same notation as in Section 2, it will be suffice to solve the following system

$$\begin{cases} (\mathcal{L}\Xi)_1^\parallel = \Theta_1^\parallel, \\ (\mathcal{L}\Xi)^\perp = \Theta^\perp. \end{cases} \quad (47)$$

For convenience, we recall the expression of  $\mathcal{L}$  in the Fermi coordinate.

$$\begin{aligned} \mathcal{L} = & A^{-1} \partial_{r_1}^2 + \partial_{z_1}^2 + \frac{1}{2} \frac{\partial_{z_1} A}{A} \partial_{z_1} - \frac{1}{2} \frac{\partial_{r_1} A}{A^2} \partial_{r_1} \\ & + r^{-1} (\partial_r r_1 \partial_{r_1} + \partial_r z_1 \partial_{z_1}) + 1 - 3u^2. \end{aligned}$$

Let us first compute the action of  $\mathcal{L}$  on functions of the form  $\xi(r_1) \eta^+ H'(z_1)$ . Using (12) we get

$$\begin{aligned} \mathcal{L}(\xi \eta^+ H') = & A^{-1} \xi'' \eta^+ H' + \left[ -\frac{1}{2} \frac{\partial_{r_1} A}{A^2} + \frac{1}{r(1+f'^2)B} \right] \xi' \eta^+ H' \\ & + \left[ \frac{1}{2} \frac{\partial_{z_1} A}{A} - \frac{f'}{r\sqrt{1+f'^2}} \right] \xi \eta^+ H'' \\ & + [H''' + (1-3u^2)H'] \xi \eta^+ \\ & + 2A^{-1} \xi' \partial_{r_1} \eta^+ H' + A^{-1} \xi \partial_{r_1}^2 \eta^+ H' + \xi \partial_{z_1} \eta^+ H'' + \xi \partial_{z_1}^2 \eta^+ H' \\ & + \left[ -\frac{1}{2} \frac{\partial_{r_1} A}{A^2} + \frac{1}{r(1+f'^2)B} \right] \xi \partial_{r_1} \eta^+ H' \\ & + \left[ \frac{1}{2} \frac{\partial_{z_1} A}{A} - \frac{f'}{r\sqrt{1+f'^2}} \right] \xi \partial_{z_1} \eta^+ H'. \end{aligned}$$

The coefficient before  $\xi'$  has the estimate

$$-\frac{1}{2} \frac{\partial_{r_1} A}{A^2} + \frac{1}{r(1+f'^2)B} = \frac{1}{r} + O(r^{-3}).$$

Next we shall estimate the coefficient before  $\xi$ . Recall that

$$\frac{1}{2} \frac{\partial_{z_1} A}{A} - \frac{f'}{r\sqrt{1+f'^2}} = O(r^{-6}).$$

We also have

$$\begin{aligned} H''' + (1 - 3u^2) H' &= 3(H^2 - u^2) H' \\ &= 3(H + u)(H - u) H'. \end{aligned}$$

Since

$$\begin{aligned} u - H &= H(z_1 - h(r_1)) - H(z_1) + O(r^{-2-\alpha}) \\ &= -h(r_1) H' + O(r^{-2-\alpha}) = O(r^{-2-\alpha}), \end{aligned}$$

hence  $H''' + (1 - 3u^2) H' = O(r^{-2-\alpha})$ . As for those terms involving derivatives of  $\eta^+$ , they could be estimated by  $O(r^{-2-\alpha})$ . Combining all the above estimate, we get

$$\mathcal{L}(\xi \eta^+ H') = A^{-1} \xi'' \eta^+ H' + \left( \frac{1}{r_1} + O(r_1^{-3}) \right) \xi' \eta^+ H' + O(r^{-2-\alpha}) \xi.$$

Introduce the operator

$$K_1 : \xi \rightarrow \int_{\mathbb{R}} \mathcal{L}(\xi \eta^+ H') \eta^+ H' dz_1.$$

It will be important to analyze the mapping property of  $K_1$ . Let us consider the equation

$$K_1 \xi = \omega. \tag{48}$$

This is a second order ODE. Suppose  $\vartheta_1, \vartheta_2$  are two linearly independent solutions of the corresponding homogeneous equation:  $K_1 \vartheta_i = 0$ . We could assume  $\vartheta_1$  is bounded near 0. Let  $\chi$  be a cutoff function equals 0 in  $(0, 1)$  and equals to 1 in  $(2, +\infty)$ . Then using the variation of parameter formula, one could show that  $K_1$  is an isomorphism from the space  $\mathcal{S}_1 \oplus \text{span}\{\chi \vartheta_2\}$  to  $\mathcal{S}_2$ . Here a function  $p \in \mathcal{S}_1$  iff  $p \in C^{2,\alpha}(\mathbb{R})$  and

$$\left\| (1+r_1)^2 p \right\|_{L^\infty} + \left\| (1+r_1)^3 p' \right\|_{L^\infty} + \max_s \left\| (1+r_1)^4 p'' \right\|_{C^{0,\alpha}(\overline{B_1(s)})} < +\infty,$$

and a function  $p \in \mathcal{S}_2$  iff  $p \in C^{0,\alpha}(\mathbb{R})$  and

$$\max_s \left\| (1+r_1)^4 p \right\|_{C^{0,\alpha}(\overline{B_1(s)})} < +\infty.$$

Define the operator

$$K_2 : \xi \rightarrow [\mathcal{L}(\xi \eta^+ H' + [\xi \eta^+ H']^s)]^\perp.$$

To solve (47), it suffices to find solution  $(\xi, \psi)$  to the system:

$$\begin{cases} K_1 \xi = \omega - \int_{\mathbb{R}} \eta^+ H' \mathcal{L} \psi dz_1, \\ (\mathcal{L} \psi)^\perp = \varphi - K_2 \xi. \end{cases} \quad (49)$$

Let  $\psi_0$  be the solution of

$$(\mathcal{L} \psi_0)^\perp = -K_2(\chi \vartheta_2),$$

satisfying  $\int_{\mathbb{R}} X^*(\psi_0 \eta^+) H' dz_1 = 0$ . Note that at infinity,  $\vartheta_1$  behaves like  $O(\ln r)$  or  $O(1)$ . Hence  $K_2 \vartheta_1$  behaves like  $O(r^{-2-\alpha})$ . This implies that  $\psi_0 = O(r^{-2-\alpha})$ . Thus  $\int_{\mathbb{R}} X^*(\mathcal{L} \psi_0 \eta^+) H' dz_1 = O(r^{-4})$ . Thanks to these estimates, one then could use a fixed point argument to get a pair of solution  $(\xi, \psi)$  for (49) with the form

$$\begin{cases} \xi = c_1 \chi \vartheta_2 + \xi_0, \\ \psi = c_1 \psi_0 + \psi, \end{cases}$$

for some constant  $c_1$ , where  $\xi_0 \in \mathcal{S}_1$  and  $\bar{\psi} \in S_1$ . (Note that  $\psi_0$  does not belong to  $S_1$ ). Then we get a corresponding solution  $\Xi$  to (47). We emphasize that at this stage, we still don't know whether the solution  $\Xi$  belongs to  $S_1$ , due to the non-decaying part

$$\hat{\Xi} := \chi(r_1) \vartheta_2(r_1) \eta^+ H'(z_1) + [\chi(r_1) \vartheta_2(r_1) \eta^+ H'(z_1)]^s + \psi_0. \quad (50)$$

**Step 2.** Investigate the homogeneous equation

$$\mathcal{L} \zeta = 0. \quad (51)$$

Firstly, we wish to find a nontrivial solution to (51). To achieve this, we use the fact that  $\vartheta_1$  is in the kernel of  $K_1$ . A perturbation argument could be applied similarly as in step 1 to get a function  $\Xi_0$  solving  $\mathcal{L} \Xi_0 = 0$ , where  $\Xi_0$  is around  $\eta^+ \vartheta_1(r_1) H'(z_1) + [\eta^+ \vartheta_1(r_1) H'(z_1)]^s$ . We could also assume  $\Xi_0$  satisfying

$$\int_{\mathbb{R}} X^*(\Xi_0 \eta^+) H'(z_1) dz_1 = \vartheta_1(r_1) + \delta \vartheta_2(r_1) + O(r_1^{-\alpha}),$$

for certain  $\delta \in \mathbb{R}$  and  $\alpha > 0$ .

Secondly we show the solution  $\Xi_0$  is in some sense unique. For this purpose, let us assume  $\Xi'_0$  is another function solves  $\mathcal{L} \Xi'_0 = 0$  and

$$\int_{\mathbb{R}} X^*(\Xi'_0 \eta^+) H'(z_1) dz_1 = \vartheta_1(r_1) + \delta' \vartheta_2(r_1) + O(r_1^{-\alpha}),$$

for certain  $\delta' \in \mathbb{R}$  and  $\alpha > 0$ . Then the function  $g := \Xi'_0 - \Xi_0$  solves  $\mathcal{L} g = 0$  and

$$\int_{\mathbb{R}} X^*(g \eta^+) H'(z_1) dz_1 = (\delta' - \delta) \vartheta_2(r_1) + O(r_1^{-\alpha}). \quad (52)$$

We claim that  $g = 0$ . Indeed, writing  $g$  as  $\eta^+ \xi(r_1) H'(z_1) + [\eta^+ \xi(r_1) H'(z_1)]^s + \varphi$ , where

$$\int_{\mathbb{R}} X^* (\eta^+ \varphi) H'(z_1) dz_1 = 0$$

and  $\varphi = O(r^{-\alpha})$  for some  $\alpha > 0$ . We have

$$\begin{cases} K_1 \xi = \int_{\mathbb{R}} \eta^+ H' \mathcal{L} \varphi dz_1, \\ (\mathcal{L} \varphi)^\perp = -K_2 \xi. \end{cases}$$

Recall that in the Fermi coordinate with respect to  $f_{k,b}$ ,

$$\begin{aligned} & \int_{\mathbb{R}} X^* (\eta^+ H'_1 \mathcal{L} \varphi) dz_1 \\ &= - \int_{\mathbb{R}} X^* (\eta^+ H'_1) \left[ A^{-1} \partial_{r_1}^2 + \frac{1}{2} \frac{\partial_{z_1} A}{A} \partial_{z_1} - \frac{1}{2} \frac{\partial_{r_1} A}{A^2} \partial_{r_1} \right] \varphi^* dz_1 \\ & - \int_{\mathbb{R}} X^* (\eta^+ H'_1) r^{-1} [\partial_{r_1} \varphi^* \partial_r r_1 + \partial_{z_1} \varphi^* \partial_r z_1] dz_1 \\ & + \int_{\mathbb{R}} [-X^* (\eta^+ H'_1) \partial_{z_1}^2 \varphi^* + X^* (\eta^+ H'_1 (3\bar{u}^2 - 1) \varphi)] dz_1. \end{aligned}$$

Hence  $\int_{\mathbb{R}} X^* (\eta^+ H'_1 \mathcal{L} \varphi) dz_1 = O(r^{-2-\alpha})$ . By (52), we could write  $\xi = \beta \vartheta_2(r_1) + \rho$ , where  $\beta \in \mathbb{R}$  and  $\rho(\cdot) = O(r_1^{-\alpha})$ . By the mapping property of the operator  $K_1$  (note that  $\vartheta_1$  is in the kernel of  $K_1$ ), for some  $\sigma \in (0, 1)$ ,

$$|\beta| + \|(1 + r_1^\alpha) \rho\|_{C^{2,\sigma}} \leq C \left\| \left( (1 + r_1^{2+\alpha}) \int_{\mathbb{R}} \eta^+ H' \mathcal{L} \varphi dz_1 \right) \right\|_{C^{0,\sigma}}. \quad (53)$$

On the other hand, by the equation  $(\mathcal{L} \varphi)^\perp = -K_2 \xi$ ,

$$\mathcal{L} \varphi = -K_2 \xi + (\mathcal{L} \varphi)^\parallel.$$

Note that due to the orthogonality condition,  $(\mathcal{L} \varphi)^\parallel$  is small compared to  $\varphi$ . Hence by the a priori estimate of the operator  $\mathcal{L}$ , suitable weighted norm of  $\varphi$  could be controlled by

$$o(|\beta|) + o\|(1 + r_1^\alpha) \rho\|_{C^{2,\sigma}}.$$

This together with (53) yields that  $g = 0$ , which implies  $\Xi'_0 = \Xi_0$ . Hence the operator  $\mathcal{L} : S_1 \oplus \mathcal{D} \rightarrow S_2$  has at most one dimensional kernel.

**Step 3.** To finish the proof, it remains to show that the solutions  $\Xi$  in step 1 and  $\Xi_0$  in step 2 indeed belong to the space  $S_1 \oplus \mathcal{D}$ .

Recall that the deficiency space  $\mathcal{D}$  is spanned by  $\gamma_1$  and  $\gamma_2$ . Consider the function  $\mathcal{L} \gamma_i, i = 1, 2$ . We know that  $\mathcal{L} \gamma_2 \in S_2$ . Hence by Step 1, one could find a solutions  $g_i$  satisfy

$$\mathcal{L} g_i = \mathcal{L} \gamma_i.$$

Note that

$$g_i - c_i \hat{\Xi} \in S_1 \quad (54)$$

for some constants  $c_i$ , where the function  $\hat{\Xi}$  is a non-decaying term introduced in (50).

Hence by Step 2 and the asymptotic behavior of  $g_i$  and  $\gamma_i$ ,  $g_i - \gamma_i = d_i \Xi_0$ ,  $i = 1, 2$ , for some constants  $d_i$ . Then by (54),

$$c_i \hat{\Xi} - \gamma_i = d_i \Xi_0, i = 1, 2.$$

It follows that  $\hat{\Xi} - k_{1,1}\gamma_1 - k_{1,2}\gamma_2 \in S_1$  and  $\bar{\Xi}_0 - k_{2,1}\gamma_1 + k_{2,2}\gamma_2 \in S_1$ , for some constants  $k_{i,j}$ ,  $i, j = 1, 2$ . The proof is completed. ■

Having proved the Fredholm property, we proceed to show that the operators involved are real analytic.

**Lemma 21** *The map  $N$  is real analytic.*

**Proof.** This follows from the fact that  $\Delta$  is a linear operator and the function  $u^3 - u$  is a real analytic function. Note that the subtle point here is that the definition of  $N$  involves the diffeomorphisms  $\Phi_{s,t}$ ,  $\Psi_{s,t}$ . These maps are certainly not real analytic with respect to the  $r, z$  variables, since there is a cutoff function appeared in their definition. However, these diffeomorphisms are indeed real analytic with respect to the parameters  $s$  and  $t$ , which could be seen from the explicit expression (11) of the Fermi coordinate (Notice that  $f$  depends analytically on  $k$  and  $b$ ). Indeed, for a point  $p = (r, z)$ , by definition,

$$\Psi_{s,t}(p) = \eta p_{s,t} + (1 - \eta)p.$$

Recall that the Fermi coordinate of  $p_{s,t}$  with respect to  $f_{k+s,b+t}$  is equal to  $(r_1, z_1)$ , which is the Fermi coordinate of  $p$  with respect to the curve  $f_{k+s,b+t}$ . Hence we have the relations

$$\begin{cases} r = r_1 - \frac{z_1 f'_{k,b}}{\sqrt{1+(f'_{k,b})^2}}, \\ z = f_{k,b}(r_1) + \frac{z_1}{\sqrt{1+(f'_{k,b})^2}}, \end{cases}$$

and

$$\begin{cases} \bar{r} = r_1 - \frac{z_1 f'_{k+s,b+t}}{\sqrt{1+(f'_{k+s,b+t})^2}}, \\ \bar{z} = f_{k+s,b+t}(r_1) + \frac{z_1}{\sqrt{1+(f'_{k+s,b+t})^2}}. \end{cases}$$

The real analyticity follows from these relations. ■

**Definition 22** *A solution  $u$  is nondegenerate, if and only if the linearized operator  $\mathcal{L} : S_1 \oplus \mathcal{D} \rightarrow S_2$  is surjective.*

By the results of [2] and [12], nondegenerate two-end solutions do exist.

**Proposition 23** *The set of solutions to (2) satisfying (45) has a structure of real analytic variety of formal dimension 1. Furthermore, if a solution  $u$  is nondegenerate, then locally around  $u$ , the solution set is a one dimensional real analytic manifold.*

**Proof.** The function  $u_{s,t,\psi}$  is a solution of the Allen-Cahn equation, if and only if

$$N(s, t, \psi) = 0. \quad (55)$$

Since  $N(0, 0, 0) = 0$ , we write equation (55) in the form

$$DN(0, 0, 0)(s, t, \psi) + \int_0^1 [DN(ls, lt, l\psi) - DN(0, 0, 0)](s, t, \psi) dl = 0.$$

Then the result follows from the fact that  $N$  is real analytic and of Fredholm index 1 and the real analytic implicit function theorem (for example, see [5]). ■

## 4 Analysis of solutions on the boundary of the moduli space

Two different types of two-end solutions to equation (2) have been constructed using Lyapunov-Schmidt reduction method in [2] and [12]. Let us briefly describe these solutions. The first type of solutions is constructed in [2] and has the property that their nodal curves are close to suitable scaling of a solution to the Toda system. We call them Toda type solutions. The growth rate of these solutions is close to  $\sqrt{2}$  (but greater than  $\sqrt{2}$ ). The second class of solutions are those constructed in [12]. Their nodal sets are close to the catenoids, which we know are described by the function  $\varepsilon r = \cosh(\varepsilon z)$ , where  $\varepsilon$  is a small parameter. We call them catenoid type solution. The growth rate of these solutions are of the order  $\varepsilon^{-1}$ , hence tends to infinity as  $\varepsilon \rightarrow 0$ .

As we discussed in Section 1, we expect that the moduli space of two-end solutions is diffeomorphic to  $\mathbb{R}$ . In particular, we expect that there exists a one-parameter family of solutions, at one end of this family (the “boundary” of the moduli space), the solutions should be the Toda type solutions, while on the other end of the moduli space, the solutions should be the catenoid type solutions.

In this section, we would like to analyze the solutions near the boundary of the moduli space. Our purpose is to prove that if  $\mathcal{P}_u$  (recall that  $\mathcal{P}_u$  is the intersection point of the nodal set of  $u$  with the coordinate axes) is on the  $z$  axis and  $|\mathcal{P}_u|$  is large, then the growth rate of  $u$  is close to  $\sqrt{2}$ . (with additional efforts, one could also show that  $u$  is actually a Toda type solution, but the proof of Theorem 1 don’t need this fact.) We shall also show that if  $\mathcal{P}_u$  is on the  $r$  axis and  $|\mathcal{P}_u|$  is large then  $u$  is a catenoid type solution.

## 4.1 Analysis of Toda type solutions

We shall first analyze the solutions whose nodal set has two components which are very far away from each other. We expect that these solutions are Toda type. Our main result here is

**Proposition 24** *Let  $u$  be two-end solution. Suppose  $\mathcal{P}_u$  is on the  $z$  axis and  $|\mathcal{P}_u|$  is large. Then the growth rate of  $u$  is close to  $\sqrt{2}$ .*

We use  $q(\cdot) = q_\varepsilon(\cdot)$  to denote the solution of the following Toda equation:

$$\mathbf{c}_0 q'' + \frac{\mathbf{c}_0}{r} q' - \mathbf{c}_1 e^{-2\sqrt{2}q} = 0, \quad q'(0) = 0. \quad (56)$$

Observe that explicitly,

$$q_\varepsilon(r) = \mathbf{q}(\varepsilon r) - \frac{\sqrt{2}}{2} \ln \varepsilon$$

for some  $\varepsilon > 0$ , where  $\mathbf{q}(r) = \frac{1}{2\sqrt{2}} \ln \frac{(1+ar^2)^2}{8}$  with  $a = \frac{2\sqrt{2}\mathbf{c}_1}{\mathbf{c}_0}$ . In particular,  $q_\varepsilon(r) - \sqrt{2} \ln r - C_\varepsilon$  tends to 0 as  $r$  tends to infinity with  $C_\varepsilon$  a constant depending on  $\varepsilon$ . In the sequel, we choose  $\varepsilon$  such that  $q(0) = |\mathcal{P}_u|$ .

We shall follow similar notations as that of Section 2. For example, the nodal curve of  $u$  in the upper half plane will be the graph of function  $f$ . The Fermi coordinate with respect to the graph of  $f$  will be denoted by  $(r_1, z_1)$ . We also have the cutoff functions  $\eta, \eta^+$ , and the solution  $u$  will be written the form  $u = \bar{u} + \phi$ , where  $\bar{u}$  is an approximate solution:

$$\bar{u} = \mathcal{H}_1 + \mathcal{H}_1^s + 1,$$

with the function  $\mathcal{H}_1$  defined similarly as that of Section 2 using the cutoff function  $\eta$  and the heteroclinic solution  $H$ .

The main idea of the proof is to compare  $f$  with the solution  $q_\varepsilon$  of (56), by analyzing the equation satisfied by  $f$ . The main step will be establishing suitable decay estimate for the function  $\phi$ , as we have already done in Section 2. Our starting point is the fact that if  $\mathcal{P}_u$  is on the  $z$  axis and  $|\mathcal{P}_u|$  is large, then  $\|f'\|_{L^\infty((0,+\infty))}$  will be small. This follows from an application of the balancing formula.

We shall get a  $L^\infty$  estimate for  $\phi$ . In the following,  $\|\cdot\|_\infty$  stands for the  $L^\infty((0,+\infty))$  norm. To simplify the notations, we only consider the case that the radius of the Fermi coordinate is large enough.

**Lemma 25** *Suppose  $u$  satisfies the assumption of Proposition 24. Then*

$$\|\phi\|_\infty \leq C \left\| \frac{f'^2}{r^2} \right\|_\infty + C e^{-2\sqrt{2}f(0)}.$$



**Proof.** We only sketch the proof, since many computations will be similar to that of Section 2.

Recall that  $\phi$  satisfies the equation

$$L\phi = [E(\bar{u})]^\parallel + [E(\bar{u})]^\perp + P(\phi). \quad (57)$$

$P(\phi)$  is a higher order term of  $\phi$ , and  $[E(\bar{u})]^\parallel = E(\bar{u})_1^\parallel + E(\bar{u})_2^\parallel$  is also small compared to  $\phi$ . As in Section 2,

$$\begin{aligned} [E(\bar{u})]^\perp &= [E(\mathcal{H}_1^s)]^\perp + O(f'^2) + O(r^{-2}f'^2) + O(h''f'') + O(h'r^{-1}) \\ &\quad + O(h'^2) + O(h'^2) + O(h'f'') + O(h'f''') + O(e^{-\sqrt{2}D}). \end{aligned} \quad (58)$$

Project equation (57) onto  $\eta^+ H_1'$ , we could show

$$\begin{aligned} &\frac{\mathbf{c}_0 f''}{(1+f'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 h''}{1+f'^2} + \frac{\mathbf{c}_0 f'}{r_1 \sqrt{1+f'^2}} - (1+o(1)) \mathbf{c}_1 e^{-\sqrt{2}D} \\ &= O(h''f'') + O(h'^2) + O(h'f'') + O(h'r^{-1}) + O(h'f''') \\ &\quad + O\left(\|\phi^*(r_1, \cdot)\|_\infty^2\right) + O\left(\|\phi^*(r_1, \cdot)\|_\infty e^{-\sqrt{2}f(r_1)}\right) + O\left(\|\phi^*(r_1, \cdot)\|_\infty \|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty\right) \\ &\quad + O\left(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty e^{-\sqrt{2}f(r_1)}\right) + O\left(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty^2\right) + O\left(\|\phi^*(r_1, \cdot)\|_\infty \|\partial_{r_1}^2 \phi^*(r_1, \cdot)\|_\infty\right) \\ &\quad + O(f'' \|\partial_{z_1} \phi^*(r_1, \cdot)\|_\infty) + O(f'' \|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty) + O(f''' \|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty) \\ &\quad + O(r^{-1} \|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty) + O(r^{-1} f' \|\partial_{z_1} \phi^*(r_1, \cdot)\|_\infty). \end{aligned} \quad (59)$$

Insert this into (58), we obtain

$$\begin{aligned} \left| [E(\bar{u})]^\perp \right| &= [E(\mathcal{H}_1^s)]^\perp + O(r^{-2}f'^2) + O(h'r^{-1}) \\ &\quad + O(h'^2) + O(h'f''') + O(e^{-\sqrt{2}D}) \\ &\quad + O(h'h'') + O(hf''') + O(h'^2) \\ &\quad + O\left(\|\phi^*(r_1, \cdot)\|_\infty^2\right) + O\left(\|\partial_{r_1} \phi^*(r_1, \cdot)\|_\infty^2\right) \\ &\quad + O\left(\|\partial_{z_1} \phi^*(r_1, \cdot)\|_\infty^2\right). \end{aligned}$$

Then the a priori estimate of the operator  $L$  tells us that

$$\|\phi\|_\infty \leq C \left\| \frac{f'^2}{r^2} \right\|_\infty + C e^{-2\sqrt{2}f(0)}.$$

■

Our next aim is to estimate the  $L^\infty$  norm of  $\frac{f'(r)}{r}$ .

**Lemma 26** *Let  $\varepsilon$  be introduced above, then*

$$\left\| \frac{f'}{r} \right\|_\infty + \|f''\|_\infty \leq C\varepsilon^2.$$

**Proof.** The starting point of the proof is still equation (59). Applying Lemma 25, we infer from (59) that

$$\begin{aligned} & \frac{\mathbf{c}_0 f''}{(1 + f'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 f'}{r_1 (1 + f'^2)^{\frac{1}{2}}} - \mathbf{c}_1 e^{-\sqrt{2}D} \\ &= O(\varepsilon^2). \end{aligned} \quad (60)$$

Let  $t_0$  be a point where

$$\frac{f'(t_0)}{t_0} = \left\| \frac{f'}{r} \right\|_{\infty}.$$

This point exists because  $\frac{f'}{r} \rightarrow 0$ , as  $r \rightarrow +\infty$ .

At the point  $t_0$ , by (60),  $f$  will satisfy

$$\begin{aligned} & \mathbf{c}_0 f''(t_0) + \mathbf{c}_0 \frac{f'(t_0)}{t_0} - \mathbf{c}_1 e^{-\sqrt{2}D(t_0)} \\ &= O(\varepsilon^2). \end{aligned} \quad (61)$$

We claim

$$f''(t_0) \geq 0. \quad (62)$$

Indeed, if this is not true, then due to the fact that  $f''(0) \geq 0$ , there will be another point  $t_1 < t_0$  such that  $f''(t_1) = 0$ , and  $f''(t) < 0$ ,  $t \in (t_1, t_0)$ . Since  $f' \geq 0$ , we find

$$\left( \frac{f'(r)}{r} \right)' = \frac{f'' - r^{-1}f'}{r} < 0, \quad r \in (t_1, t_0).$$

This contradicts with the fact that  $\frac{f'(t_0)}{t_0} = \left\| \frac{f'}{r} \right\|_{\infty}$ . From (61) and (62), it may be concluded that

$$\begin{aligned} \frac{f'(t_0)}{t_0} &= \frac{\mathbf{c}_1}{\mathbf{c}_0} e^{-\sqrt{2}D(t_0)} - f''(t_0) + O(\varepsilon^2) \\ &\leq C e^{-2\sqrt{2}f(0)} \leq C \varepsilon^2. \end{aligned} \quad (63)$$

Here we have used the fact that  $D(t_0) \geq 2f(0)$ . This proves estimate for the  $L^\infty$  norm of  $\frac{f'}{r}$ . The estimate for  $f''$  is a direct consequence of (63) and (61). ■

Let  $p = f + \sqrt{1 + f'^2}h$ . Next we show that  $p$  is indeed close to the solution  $q$  of the Toda equation in a large interval. Set  $b = b_\varepsilon = \left\lfloor \frac{\ln \varepsilon}{\varepsilon} \right\rfloor$ .

**Lemma 27** *There exists  $\alpha > 0$ , such that*

$$|p(r) - q(r)| \leq C \varepsilon^\alpha, \quad r \in (0, b).$$

**Proof.** Denote by  $\mathbf{e}$  the function  $p(\varepsilon^{-1}r) + \frac{\sqrt{2}}{2} \ln \varepsilon - \mathbf{q}(r)$ . Then  $\mathbf{e}(0) = O(\varepsilon^2)$  and  $\mathbf{e}'(0) = 0$ . By the previous lemmas,  $\|\phi\|_\infty \leq \varepsilon^2$ . Using this fact, we see that the function  $p = \sqrt{1 + f'^2}h + f$  satisfies the equation

$$\frac{\mathbf{c}_0 p''}{(1 + p'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0 p'}{r_1 (1 + p'^2)^{\frac{1}{2}}} - \mathbf{c}_1 e^{-\sqrt{2}D} = O(\varepsilon^{2+\alpha}).$$

In particular, this combined with the fact that  $D - 2f = O(\varepsilon^\alpha)$  yields

$$p'' + \frac{p'}{r_1} - e^{-2\sqrt{2}p} = O(\varepsilon^{2+\alpha}).$$

From this equation, we infer that in the region where  $\mathbf{e}$  is  $o(1)$ ,  $\mathbf{e}$  satisfies

$$\mathbf{e}'' + \frac{\mathbf{e}'}{r} - e^{-2\sqrt{2}\mathbf{q}}\mathbf{e} = O(\mathbf{e}^2) + O(\varepsilon^\alpha).$$

The conclusion of the lemma then follows from the variation of parameters formula. ■

By definition  $q(r) = \mathbf{q}(\varepsilon r) - \frac{\sqrt{2}}{2} \ln \varepsilon$ , hence  $rq'(r) = \varepsilon r \mathbf{q}'(\varepsilon r)$ , which implies that

$$bq'(b) = \sqrt{2} + o(1).$$

Notice that in Lemma 27, we actually could also estimate  $|\mathbf{e}'| \leq C\varepsilon^\alpha$ . From this, we infer that

$$bf'(b) = \sqrt{2} + o(1). \quad (64)$$

Now we are in a position to prove the main result of this section.

**Proof of Proposition 24.** Similar arguments as before yields

$$|\phi(r, z)| \leq C \frac{1}{1+r^2} + Ce^{-D(r)}.$$

Equation (59) then becomes

$$\frac{p''}{(1+p'^2)^{\frac{3}{2}}} + \frac{p'}{r_1(1+p'^2)^{\frac{1}{2}}} = (1+o(1))e^{-\sqrt{2}D(r_1)} + O(r_1^{-3}).$$

Integrating from  $t_0$  to  $t_1$  leads to

$$\frac{t_1 p'(t_1)}{(1+p'(t_1)^2)^{\frac{1}{2}}} - \frac{t_0 p'(t_0)}{(1+p'(t_0)^2)^{\frac{1}{2}}} = (1+o(1)) \int_{t_0}^{t_1} e^{-\sqrt{2}D(s)} s ds + O(t_0^{-1}).$$

This together with (64) tells us that

$$p'(r)r \geq \sqrt{2} + o(1) \text{ for } r > b.$$

Let  $\delta > 0$  be a fixed small constant. We claim that when  $\varepsilon$  is small,

$$p'(r)r \leq \sqrt{2} + \delta \text{ for } r > b. \quad (65)$$

Indeed, suppose  $(b, b^*)$  is the maximal interval where  $p'(r)r \leq \sqrt{2} + \delta$ . Then in this interval, elementary geometrical facts implies that

$$\begin{aligned} D(r) &\geq \sqrt{2}|\ln \varepsilon| + 2\mathbf{q}(b) + 2\left(\sqrt{2} + o(1)\right) \int_b^r \frac{ds}{s} - C \\ &= \sqrt{2}|\ln \varepsilon| + 2\mathbf{q}(b) + 2\left(\sqrt{2} + o(1)\right) (\ln r - \ln b) - C. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{b^* p'(b^*)}{\left(1 + p'(b^*)^2\right)^{\frac{1}{2}}} - \frac{b p'(b)}{\left(1 + p'(b)^2\right)^{\frac{1}{2}}} &= (1 + o(1)) \int_b^{b^*} e^{-\sqrt{2}D(s)} s ds + O(b^{-1}) \\ &= O(b^{-1}). \end{aligned}$$

This implies that when  $\varepsilon$  is small,  $b^* = +\infty$ . Applying (65), we finally get  $\lim_{r \rightarrow +\infty} p'(r) r = \sqrt{2} + o(1)$ . The proof is thus completed. ■

## 4.2 Uniqueness of catenoid type solutions

Beside the planes, catenoid is the first example of embedded minimal surfaces with finite total curvature; it is rotationally symmetric with respect to its axis and the only minimal surface of revolution (up to a homothety). In the  $(r, z)$  coordinate, the one-parameter family of catenoids  $\mathcal{C}_\varepsilon$  could be represented by the function

$$\varepsilon r = \cosh(\varepsilon z),$$

with  $\varepsilon > 0$  being the parameter. As we mentioned before, in [12], for each  $\varepsilon$  sufficiently small, a solution  $u_\varepsilon$  of the Allen-Cahn equation is constructed. The nodal set of  $u_\varepsilon$  is close to the catenoid  $\mathcal{C}_\varepsilon$ . In particular,  $\mathcal{P}_{u_\varepsilon}$  is on the  $r$  axis,  $|\mathcal{P}_{u_\varepsilon}|$  is of the order  $O(\varepsilon^{-1})$ , and the growth rate of  $u_\varepsilon$  is also of the order  $O(\varepsilon^{-1})$ .

In this section, we wish to prove that this one-parameter family of solutions  $u_\varepsilon$  is unique. This is the content of the following

**Proposition 28** *Let  $u$  be a two-end solution of the Allen-Cahn equation. Suppose  $\mathcal{P}_u$  is on the  $r$  axis and  $|\mathcal{P}_u|$  is large. Then there exists a small  $\varepsilon' > 0$  such that  $u = u_{\varepsilon'}$ .*

The nodal curve  $\mathcal{N}_u$  in  $\mathbb{E}^+$  could be written as

$$\{(r, z) : z = f(r)\},$$

where  $f$  is a function

$$f : [|\mathcal{P}_u|, +\infty) \rightarrow \mathbb{R}.$$

Note that by the results in the previous section,  $f$  is asymptotic to  $c_1 \ln r + c_2$  as  $r \rightarrow +\infty$  for some constants  $c_1$  and  $c_2$ . One could also write  $\mathcal{N}_u \cap \mathbb{E}^+ = \{(r, z) : r = g(z)\}$  for some function  $g : [0, +\infty) \rightarrow \mathbb{R}$ .

For convenience we introduce the parameter  $\varepsilon = |\mathcal{P}_u|^{-1}$ . Observe that by the assumption of Proposition 28,  $\varepsilon$  is small, thus by the validity of De Giorgi conjecture in  $\mathbb{R}^3$ , locally around the nodal curve,  $u$  looks like the heteroclinic solution.

Let  $l$  be a large but fixed constant. As a preliminary step, we would like to get some rough information about the slope of the function  $f$ .

**Lemma 29** *We have*

$$f'(r) > \frac{C}{l} + o(1), \quad r \in [\varepsilon^{-1}, l\varepsilon^{-1}].$$

**Proof.** Let  $X = (0, 0, 1)$  be a constant vector field. We have

$$\int_0^{l\varepsilon^{-1}} \left[ \frac{1}{2} u_r^2 + F(u) \right] r dr \geq Cl\varepsilon^{-1}.$$

On the other hand,

$$\begin{aligned} & \int_{\partial\Omega \cap \{z > 0\}} \left\{ \left[ \frac{1}{2} u_r^2 + F(u) \right] X \cdot v - (\nabla u \cdot X) (\nabla u \cdot v) \right\} dS \\ & \sim f'(l\varepsilon^{-1}) l\varepsilon^{-1} + o(1) l\varepsilon^{-1}. \end{aligned}$$

Combine these two estimates and use the balancing formula, we get the desired result. ■

To get more precise information, we again need to work in the Fermi coordinate  $(s, t)$  around the nodal curve  $\mathcal{N}_u$ , where  $s$  is the signed distance to  $\mathcal{N}_u$  and  $t$  is a parametrization of  $\mathcal{N}_u$ .

We slightly abuse the notation and still use  $\mathcal{B}_u$  to denote the maximal domain where the Fermi coordinate is well-defined. Similarly as before, let  $\eta$  be a cutoff function supported in  $\mathcal{B}_u$ , with  $\nabla\eta$  supported near the boundary of  $\mathcal{B}_u$ . Introduce the approximate solution

$$\bar{u}(r, z) = \eta\mathcal{H} + (1 - \eta) \frac{\mathcal{H}}{|\mathcal{H}|},$$

where  $\mathcal{H}$  is defined through  $X^*\mathcal{H}(s, t) = H(s - h(t))$ . Here  $X$  is the map  $(s, t) \rightarrow (r, z)$ . Write  $u = \bar{u} + \phi$ . The small function  $h$  is chosen such that  $\phi$  satisfies the orthogonal condition

$$\int_{\mathbb{R}} X^*(\eta\phi\mathcal{H}') ds = 0,$$

where  $X^*\mathcal{H}'(s, t) = H'(s - h(t))$ .

To analyze the solution  $u$ , we firstly study  $\mathcal{N}_u$  in the region where  $r \in (\varepsilon^{-1}, l\varepsilon^{-1})$ . In this region, many calculations will be explicit to in the Fermi coordinate  $(x, y)$  with respect to the graph of the function  $g$ , which is

$$\begin{cases} r = g(y) + \frac{x}{\sqrt{1+g'^2}}, \\ z = y - \frac{xg}{\sqrt{1+g'^2}}. \end{cases}$$

Hence  $x$  is the signed distance and  $y$  is a parametrization of the curve.

We shall estimate the  $L^\infty$  norm of the perturbation term  $\phi$ .

**Lemma 30** *Suppose  $u$  satisfies the assumption of Proposition 28 and  $\phi$  is defined above. Then*

$$\|\phi\|_\infty \leq C\varepsilon^2.$$

**Proof.** We only consider the case that the size of the Fermi coordinate is large enough. The general case could be handled using the arguments of Section 2. Again many computations here are similar as before.

We need to analyze  $E(\bar{u})$ . First of all, consider the case  $r \in (\varepsilon^{-1}, l\varepsilon^{-1})$ . In the Fermi coordinate, the error of the approximate solution has the form

$$\begin{aligned} E(\bar{u}) &= \frac{H''h'^2 - H'h''}{A} + \left( \frac{\partial_x A}{2A} + \frac{\partial_r x}{r} \right) H' \\ &\quad + \left( \frac{\partial_y A}{2A^2} - \frac{\partial_r y}{r} \right) H'h'. \end{aligned}$$

Keep in mind that  $H'$  is evaluated at  $x - h(y)$ , not  $x$ . By Lemma 29,  $|g'| \leq C$ , hence

$$\frac{\partial_r x}{r} = \frac{\frac{1}{\sqrt{1+g'^2}}}{g + \frac{x}{\sqrt{1+g'^2}}} = \frac{1}{g\sqrt{1+g'^2}} - \frac{x}{g^2(1+g'^2)} + O(g^{-3}).$$

From  $|g''| = o(1)$ , we infer that

$$\frac{\partial_x A}{2A} = -\frac{g''}{(1+g'^2)^{\frac{3}{2}}} + \frac{(g'')^2 x}{(1+g'^2)^3} + O(g''^3).$$

It follows from these expansions that the projection of  $E(\bar{u})$  onto  $\mathcal{H}'$  has the form

$$\begin{aligned} \int_{\mathbb{R}} X^*(\eta \mathcal{H}' E(\bar{u})) dx &= -\frac{\mathbf{c}_0 g''}{(1+g'^2)^{\frac{3}{2}}} + \frac{\mathbf{c}_0}{g\sqrt{1+g'^2}} \\ &\quad - h'' \int_{\mathbb{R}} \frac{H'^2}{A} + O(h'^2) + O(h'g'') \\ &\quad + O(h'g''') + O(g''^3) + O(g^{-3}) \\ &\quad + O(hg''^2) + O(hg^{-2}). \end{aligned}$$

Set  $\tilde{h}(y) = \sqrt{1+g'^2}h(y)$  and  $p_1(y) = g(y) + \tilde{h}(y)$ . Then

$$\begin{aligned}
& -\frac{g''}{1+g'^2} + \frac{1}{g} - \frac{h''}{\sqrt{1+g'^2}} \\
& = -\frac{g''}{1+g'^2} + \frac{1}{g} - \frac{\tilde{h}''}{1+g'^2} - \frac{2\tilde{h}'}{\sqrt{1+g'^2}} \left( \frac{1}{\sqrt{1+g'^2}} \right)' \\
& \quad - \frac{\tilde{h}}{\sqrt{1+g'^2}} \left( \frac{1}{\sqrt{1+g'^2}} \right)'' \\
& = -\frac{p_1''}{1+p_1'^2} + \frac{1}{p_1} + O(g''h) + O(g''h') + O(g'''h) + O(g'''h') \\
& \quad + O(g''h) + O(g''h') + O(h^2) + O(hh') + O(g^{-1}h).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sqrt{1+g'^2} \int_{\mathbb{R}} X^*(\eta \mathcal{H}' E(\bar{u})) dx &= -\frac{\mathbf{c}_0 p_1''}{1+p_1'^2} + \frac{\mathbf{c}_0}{p_1} \\
& \quad + O(g''h) + O(g''h') + O(g'''h) + O(g'''h') \\
& \quad + O(g''h) + O(g''h') + O(h^2) + O(hh') + O(g^{-1}h) \\
& \quad + O(h'^2) + O(g'^3) + O(g^{-3}) + O(h'h'').
\end{aligned}$$

On the other hand, we could also show that

$$\int_{\mathbb{R}} X^*(\eta \mathcal{H}' E(\bar{u})) dx = o(\|\phi^*(y, \cdot)\|_{C^2}) + o(\|\partial_y \phi^*(y, \cdot)\|_{C^0}) + o\left(\|\partial_y^2 \phi^*(y, \cdot)\|_{C^0}\right). \quad (66)$$

Here the norm is taken as a function of  $x$  variable. As a consequence of the above two equations, we have

$$\begin{aligned}
-\frac{p_1''}{1+p_1'^2} + \frac{1}{p_1} &= O(h'^2) + O(h'g'') + O(h'g''') + O(g''^3) \\
& \quad + O(g^{-3}) + O(hg''^2) + O(hg^{-2}) + O(h'h'') \\
& \quad + o(\|\phi^*(y, \cdot)\|_{C^2}) + o(\|\partial_y \phi^*(y, \cdot)\|_{C^0}) + o\left(\|\partial_y^2 \phi^*(y, \cdot)\|_{C^0}\right). \quad (67)
\end{aligned}$$

Next we consider the case  $r > \varepsilon^{-1}l$ . In this case, one could use the Fermi coordinate  $(r_1, z_1)$  with respect the curve  $z = f(r)$ . The analysis of  $E(\bar{u})$  in this case is almost same as that of the previous sections and we omit the details.

Now we write the equation satisfied by  $\phi$  into the form

$$L\phi = [E(\bar{u})]^\perp + [E(\bar{u})]^\parallel + P(\phi),$$

where

$$[E(\bar{u})]^\parallel = \frac{\int_{\mathbb{R}} X^*(\eta \mathcal{H}' E(\bar{u})) dx}{\int_{\mathbb{R}} \eta^2 \mathcal{H}'^2 dx} \eta \mathcal{H}' \text{ and } [E(\bar{u})]^\perp = E(\bar{u}) - [E(\bar{u})]^\parallel.$$

In terms of Fermi coordinate, in the region where  $|g'| \leq C$ ,

$$[E(\bar{u})]^\perp = O(|p_1''|^2) + O(p_1^{-2}).$$

In the region where  $|f'| \leq C$ ,  $[E(\bar{u})]^\perp = O(r_1^{-2})$ . By the a priori estimate of  $L$ , we could obtain

$$\|\phi\|_\infty \leq \|[E(\bar{u})]^\perp\|_\infty \leq C\varepsilon^2.$$

We emphasize here that  $\|[E(\bar{u})]^\perp\|_\infty$  should be estimated in the whole plane. It is also worth mentioning that the term  $O(|p_1''|^2)$  in  $[E(\bar{u})]^\perp$  should be handled using equation (67). ■

Our next aim is to show that in the interval  $(\varepsilon^{-1}, l\varepsilon^{-1})$ , the function  $r = p_1(z)$  is close to the function  $r = \varepsilon^{-1} \cosh(\varepsilon z)$ . This is the content of the following

**Lemma 31** For  $z \in (0, f(l)\varepsilon^{-1})$ ,

$$|p_1(z) - \varepsilon^{-1} \cosh(\varepsilon z)| \leq C\varepsilon.$$

**Proof.** From equation (67) and the estimate of  $\phi$ , we deduce that the function  $p_1$  satisfies

$$\frac{p_1''}{1 + p_1'^2} - \frac{1}{p_1} = O(\varepsilon^3).$$

At this stage, we introduce the scaled function  $\bar{p}_1(z) = \varepsilon p_1(\varepsilon^{-1}z)$ . Then

$$\begin{aligned} \bar{p}_1'(z) &= p_1'(\varepsilon^{-1}z), \\ \bar{p}_1''(z) &= \varepsilon^{-1} p_1''(\varepsilon^{-1}z). \end{aligned}$$

It follows that

$$\bar{p}_1'' - \frac{1 + \bar{p}_1'^2}{\bar{p}_1} = O(\varepsilon^2), \quad z \in (0, l). \quad (68)$$

Observe that the function  $\cosh z$  satisfies the equation

$$(\cosh z)'' - \frac{1 + (\cosh z)'^2}{\cosh z} = 0. \quad (69)$$

Let  $\omega(z) = \bar{p}_1(z) - \cosh z$ . Then

$$\begin{aligned} \omega(0) &= \bar{p}_1(0) - 1 = \varepsilon p_1(0) - 1 \\ &= \varepsilon(\varepsilon^{-1} + h(0)) - 1 = O(\varepsilon^2). \end{aligned}$$

We claim that  $|\omega(z)| \leq C\varepsilon^2$  for  $z \in (0, l)$ . Indeed, subtracting equation (68) with (69), we get

$$\omega'' - 2 \tanh z \omega' + \omega = O(\varepsilon^2) + O(\omega^2) + O(\omega'^2) := \psi(z).$$



Let  $\xi_1$  and  $\xi_2$  be two linearly independent solutions of the homogeneous equation:

$$\xi_i'' - 2 \tanh z \xi_i' + \xi_i = 0, i = 1, 2.$$

Explicitly, we can choose

$$\begin{aligned}\xi_1(z) &= \cosh' z = \sinh z, \\ \xi_2(z) &= \partial_\varepsilon (\varepsilon^{-1} \cosh \varepsilon z) |_{\varepsilon=1} = -\cosh z + z \sinh z.\end{aligned}$$

The Wronsky of these two solutions are

$$W(z) := \begin{vmatrix} \sinh z & \cosh z \\ -\cosh z + z \sinh z & z \cosh z \end{vmatrix} = \cosh^2 z.$$

By the variation of parameters formula,

$$\omega(z) = \xi_2(z) \int_0^z \frac{\xi_1(s) \psi(s)}{W(s)} ds - \xi_1(z) \int_0^z \frac{\xi_2(s) \psi(s)}{W(s)} ds + O(\varepsilon^2).$$

The desired estimate comes from this formula.  $\blacksquare$

**Proof of Proposition 28.** Let  $p_2 = f + \sqrt{1 + f'^2}h$ . For  $r > l\varepsilon^{-1}$ , projecting  $E(\bar{u})$  on  $\eta\mathcal{H}'$  and perform similar calculation as in Section 2, we could estimate the perturbation  $\phi$  in algebraically weighted norm (Remember that  $h$  could be controlled by  $\phi$ ). This leads to the equation:

$$\left( \frac{r_1 p_2'}{\sqrt{1 + p_2'^2}} \right)' = O\left( \frac{\varepsilon^2}{1 + \varepsilon^2 r^2} \right).$$

Introduce the scaling of  $p_2$ :

$$\bar{p}_2(r_1) = \varepsilon p_2(\varepsilon^{-1} r_1).$$

We find that in  $(l, +\infty)$ ,  $\bar{p}_2$  satisfies has the equation

$$\left( \frac{r_1 \bar{p}_2'}{\sqrt{1 + \bar{p}_2'^2}} \right)' = O\left( \frac{\varepsilon^2}{1 + r^2} \right). \quad (70)$$

From this equation, we deduce

$$\lim_{r_1 \rightarrow +\infty} r_1 \bar{p}_2'(r_1) - l \bar{p}_2'(l) = O(\varepsilon^2).$$

On the other hand, by Lemma 31,

$$l \bar{p}_2'(l) - 1 = O(\varepsilon^2).$$

Hence

$$\lim_{r_1 \rightarrow +\infty} r_1 \bar{p}_2'(r_1) = 1 + O(\varepsilon^2).$$

Consequently, the growth rate of  $u$  is equal to

$$\lim r_1 p_2'(r_1) = \frac{1}{\varepsilon} + O(\varepsilon) := \frac{1}{\varepsilon'}.$$

Obviously,  $\varepsilon' = \varepsilon + O(\varepsilon^3)$ .

Now let  $r = \varepsilon'^{-1} \cosh(\varepsilon' z)$  be the catenoid with the same growth rate as  $u$ . Then there is a solution  $u_{\varepsilon'}$  with the slope  $\varepsilon'^{-1}$  whose nodal line is close to this catenoid (the nodal line in  $\mathbb{E}^+$  decaying algebraically to a vertical small translation of the catenoid). Actually, the error could be estimated by some positive power of  $\varepsilon$ . Our aim is to show that  $u = u_{\varepsilon'}$ . To see this, we need to estimate the function  $p_1(z) - \varepsilon'^{-1} \cosh(\varepsilon' z)$  and  $p_2(r_1) - \varepsilon'^{-1} \cosh^{-1}(\varepsilon' r_1)$ .

Note that

$$\begin{aligned} \varepsilon \varepsilon'^{-1} \cosh(\varepsilon' \varepsilon^{-1} z) - \cosh z &= (1 + O(\varepsilon^2)) \cosh(1 + O(\varepsilon^2) z) - \cosh z \\ &= O(\varepsilon^2). \end{aligned}$$

Hence using Lemma 31, we obtain  $\bar{p}_1(z) - \varepsilon \varepsilon'^{-1} \cosh(\varepsilon' \varepsilon^{-1} z) = O(\varepsilon^2)$ . This implies that

$$p_1(z) - \varepsilon'^{-1} \cosh(\varepsilon' z) = O(\varepsilon).$$

Using (70), we then find that  $p_2(r_1) - \varepsilon'^{-1} \cosh^{-1}(\varepsilon' r_1) = O(\varepsilon)$ .

Therefore the nodal line of  $u$  is close to the nodal line of  $u_{\varepsilon'}$  at the order  $O(\varepsilon)$ . Then, applying the mapping property of the Jacobi operator of the catenoid, a contraction mapping argument shows that  $u = u_{\varepsilon'}$ . We refer to [19] for similar arguments in the case of four-end solutions in  $\mathbb{R}^2$ . ■

## 5 Concluding the Proof of Theorem 1

In this section, combining the results of the previous sections, we would like to finish the proof of Theorem 1.

Let  $M$  be the set of all two-end solutions to the Allen-Cahn equation with growth rate larger than  $\sqrt{2}$ . Consider a catenoid type solution  $u_\varepsilon$  arising from a largely dilated catenoid. As we mentioned in Section 3,  $u_\varepsilon$  is nondegenerate. By Proposition 23, locally around  $u_\varepsilon$ , the set of two-end solutions is a one dimensional real analytic manifold, which we denote as the image of a map  $\varrho$

$$\varrho : (-\delta, \delta) \rightarrow M,$$

for some small  $\delta > 0$ . Using the compactness result in Section 2 and by the structure theorem (19),  $\varrho$  has a global continuation:

$$\varrho : (-\delta, +\infty) \rightarrow M.$$

We claim that the growth rate of the solution  $\varrho(t)$  tends to  $\sqrt{2}$  as  $t \rightarrow +\infty$ . Indeed, as  $t \rightarrow +\infty$ ,  $\mathcal{P}_{\varrho(t)}$  could not remain bounded. (Recall that  $\mathcal{P}_{\varrho(t)}$  is the intersection of the nodal set of  $\varrho(t)$  with the  $r$  or  $z$  axis.) Otherwise by

the compactness, the image of  $\varrho$  will be a closed loop, which could not be true since the family of catenoid type solutions  $u_\varepsilon$  are not compact. Also, as  $t \rightarrow +\infty$ ,  $\mathcal{P}_{\varrho(t)}$  could not be on the  $r$  axis, this follows from the uniqueness of catenoid type solutions, Proposition 28. Therefore,  $\mathcal{P}_{\varrho(t)}$  will be on the  $z$  axis and  $|\mathcal{P}_{\varrho(t)}| \rightarrow +\infty$ , as  $t \rightarrow +\infty$ . By the analysis of Toda type solutions, Proposition 24, the growth rate of the solution  $\varrho(t)$  will go to  $\sqrt{2}$  as  $t \rightarrow +\infty$ . This finishes the proof.

## 6 Appendix

### 6.1 Monotonicity of two-end solutions

In this appendix, we sketch the proof of monotonicity of two-end solutions (Proposition 3 in Section 2) using the moving plane argument. Essentially, we follow the proof of monotonicity for four-end solutions of 2D Allen-Cahn equation in [14]. But there is a slight difference here. Namely for the two-end solutions in dimension three, to start the moving procedure at infinity, one needs to have suitable control of the asymptotic behavior of the solution (estimate (73) below), while this is not needed in the dimension two case. The asymptotic expansion we need is provided by the results of Section 2.

**Proof of Proposition 3.** Let  $u$  be a two-end solution. We first prove its monotonicity in the  $r$  direction. To use the moving plane machinery, we will work in the usual Euclidean coordinate  $(x, y, z)$ . Set  $U(x, y, z) := u(\sqrt{x^2 + y^2}, z) = u(r, z)$ .

Suppose the growth rate of  $u$  is equal to  $k > \sqrt{2}$ . Hence the asymptotic curve of its nodal line in the first quadrant is

$$z = k \ln r + b$$

for some  $b \in \mathbb{R}$ .

Let  $x_0 \geq 0$  be a parameter and define

$$\bar{U}(x, y, z) = U(x, y, z; x_0) := U(2x_0 - x, y, z).$$

Certainly  $\bar{U}(x_0, y, z) = U(x_0, y, z)$ . Note that  $\bar{U}$  actually depends on the parameter  $x_0$ .

The first step is to show that the moving plane procedure could be started at  $+\infty$ . We claim that for  $x_0$  large enough, say  $x_0 > a_0$ ,

$$\bar{U}(x, y, z; x_0) < U(x, y, z), \text{ for } x < x_0.$$

First of all, we consider the region where  $|\bar{U}|$  is not close to 1. Recall that the nodal set of  $\bar{U}$  is close to

$$z = k \ln \sqrt{(2x_0 - x)^2 + y^2} + b.$$

We have

$$\begin{aligned} & \ln \sqrt{(2x_0 - x)^2 + y^2} - \ln \sqrt{x^2 + y^2} \\ &= \frac{1}{2} \ln \left( 1 + \frac{4x_0^2 - 4x_0x}{r^2} \right). \end{aligned} \quad (71)$$

Note that for  $x < x_0$ ,  $\frac{4x_0^2 - 4x_0x}{r^2} > 0$ . Fix a small positive constant  $\epsilon$ .

If  $\frac{4x_0^2 - 4x_0x}{r^2} > \epsilon$ , then by (71),

$$\ln \sqrt{(2x_0 - x)^2 + y^2} - \ln \sqrt{x^2 + y^2} > \frac{1}{2} \ln(1 + \epsilon). \quad (72)$$

By the results of Section 2, for  $r$  large, we have

$$u(r, z) - H(z_1) = O(r^{-2}). \quad (73)$$

Here  $z_1$  is the signed distance of  $(r, z)$  to the nodal line. As a consequence

$$\begin{aligned} & \bar{U}(x, y, z; x_0) - U(x, y, z) \\ &= u\left(\sqrt{(2x_0 - x)^2 + y^2}, z\right) - u(r, z) \\ &= H\left(\left[z - k \ln \sqrt{(2x_0 - x)^2 + y^2} - b\right] \cos \theta_1\right) \\ &\quad - H\left(\left[z - k \ln \sqrt{x^2 + y^2} - b\right] \cos \theta_2\right) + O(r^{-2}), \end{aligned}$$

where  $\theta_i = O(r^{-1})$ . It follows from (72) that

$$\begin{aligned} \bar{U}(x, y, z; x_0) - U(x, y, z) &\leq H\left(z - k \ln \sqrt{(2x_0 - x)^2 + y^2} - b\right) \\ &\quad - H\left(z - k \ln \sqrt{x^2 + y^2} - b\right) + O(r^{-2}) \\ &< 0, \end{aligned}$$

for  $r$  large enough.

If  $\frac{4x_0^2 - 4x_0x}{r^2} \in (0, \epsilon)$ , then

$$\ln \sqrt{(2x_0 - x)^2 + y^2} - \ln \sqrt{x^2 + y^2} = \frac{2x_0^2 - 2x_0x}{r^2} + O\left(\frac{x_0^2(x_0 - x_0)^2}{r^4}\right). \quad (74)$$

There are two possible cases. Case 1:  $x \in (-\infty, x_0 - 1)$ . In this case, by (74),

$$\ln \sqrt{(2x_0 - x)^2 + y^2} - \ln \sqrt{x^2 + y^2} \geq \frac{x_0}{r^2}.$$

This together with (73) implies that for  $x_0$  large,  $\bar{U} < U$ . Case 2:  $x \in (x_0 - 1, x_0)$ .

In this case, by the estimate

$$\partial_r(u(r, z) - H(z_1)) = O(r^{-2}),$$

we find that (here one should also use some estimates of the Fermi coordinate)

$$\bar{U} - U \leq -C \frac{x_0^2 - x_0 x}{r^2} + O(r^{-2}(x_0 - x))$$

for certain positive constant  $C$ . Hence if we choose  $x_0$  large,  $\bar{U} < U$ . Therefore, to prove the claim, it remains to consider the region where  $|\bar{U}| \sim 1$ . Observe that for  $x_0$  large enough, in the region where  $\bar{U} \sim 1$ ,  $U$  is also close to 1. Now let  $\varphi := \bar{U} - U$ . Then in the region  $x < x_0$ ,  $\varphi$  satisfies

$$-\Delta_{(x,y,z)}\varphi + (\bar{U}^2 + \bar{U}U + U^2 - 1)\varphi = 0.$$

Since  $\limsup_{|(x,y,z)| \rightarrow +\infty} \varphi \leq 0$ , by the maximum principle,  $\varphi(x, y, z) < 0$ , for  $x < x_0$ . This proves the claim.

In the second step, we define

$$x^* = \inf \{ \bar{x} : \bar{U}(x, y, z; x_0) < U(x, y, z) \text{ for } x < x_0 \text{ and } x_0 \in (\bar{x}, a_0) \}.$$

We show that  $x^* = 0$ . To see this, we first prove that  $r^* < a_0$ . Indeed, by the first step,

$$\begin{aligned} \bar{U}(x, y, z; a_0) &< U(x, y, z) \text{ for } x < a_0, \\ \bar{U}(a_0, y, z; a_0) &= U(a_0, y, z). \end{aligned}$$

Hence by the Hopf Lemma,  $\partial_x (\bar{U}(\cdot; a_0) - U) > 0$  for  $x = a_0$ . Then standard arguments together with the asymptotic behavior of  $U$  implies that  $\bar{U}(x, y, z; x_0) < U$ , for  $x < x_0$  and  $x_0$  sufficiently close to  $a_0$ . Then one could use this type of arguments to show that  $x^* = 0$ . (The plane could be moved to the left until the inequality  $\bar{U} < U$  is violated at infinity.)

The monotonicity in  $z$  direction could be proved similarly using moving plane. This completes the proof. ■

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